Analyse Asymptotique
Perturbations singulières

Fondements méthodologiques

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Pourquoi l'analyse asymptotique ?

$$A + B\varepsilon = 0$$

1ère Approximation : $A = A_0 = 0$ Problème ?

Zéro asymptotique $\neq$ Zéro numérique

2ème Approximation : $B = 0$

Sauf si ?
$A(\varepsilon)$ et $B(\varepsilon)$

Hypothèses

$$A \text{ analytique, } A_n = \sum_{n=0}^{n} A_n \varepsilon^n$$

$$B \text{ analytique, } B_n = \sum_{n=0}^{n} B_n \varepsilon^n$$

Conclusion : $\forall n > 1,\ A_n + B_{n-1} = 0$

Problèmes asymptotiques.

$$\mathcal{L}_\varepsilon[\phi(x,\varepsilon)] = 0$$

$$\mathcal{L}_0[\phi_0(x)] = 0$$

$$\|\phi - \phi_0\| \ll 1$$

norme de la convergence uniforme :

$$\mathrm{Max}_{\mathcal{D}}|\phi_\varepsilon - \phi_0| < K\delta(\varepsilon)$$

$$\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$$

L'oscillateur linéaire

$$m\frac{\mathrm{d}^2 y^*}{\mathrm{d}t^{*2}} + \beta\frac{\mathrm{d}y^*}{\mathrm{d}t^*} + ky^* = 0$$

Conditions initiales $t^* = 0$:

$$y^*(0) = 0 \qquad m\frac{\mathrm{d}y^*}{\mathrm{d}t^*}(0) = I_0$$

$$y^*(t^*; m, \beta, k, I_0)$$

Trois raisons pour simplifier

1. Simplifier l'étude mathématique
2. Condenser les résultats
3. Esquisser une théorie des maquettes

Adimensionnalisation

Soit : $y = \dfrac{y^*}{L}$ et $t = \dfrac{t^*}{T}$

$$\frac{m}{kT^2}\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{\beta}{kT}\frac{\mathrm{d}y}{\mathrm{d}t} + y = 0$$

Conditions initiales $t = 0$:

$$y(0) = 0 \qquad \text{et} \qquad \frac{\mathrm{d}y}{\mathrm{d}t}(0) = \frac{I_0 T}{mL}$$

$$\hookrightarrow T = \frac{mL}{I_0}$$

$$\frac{I_0^2}{mL^2 k}\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{\beta I_0}{mLk}\frac{\mathrm{d}y}{\mathrm{d}t} + y = 0$$

avec $\quad y(0) = 0 \quad$ et $\quad \dfrac{\mathrm{d}y}{\mathrm{d}t}(0) = 1$

2 choix possibles : $L = \dfrac{I_0}{\sqrt{mk}}$ et $L = \dfrac{\beta I_0}{mk}$

$$\frac{\beta l_0}{mLk} \ll \frac{l_0^2}{mL^2 k} \qquad L = \frac{l_0}{\sqrt{mk}} \qquad \varepsilon = \frac{\beta}{2\sqrt{mk}}$$

$$\mathcal{L}_\varepsilon[y(t,\varepsilon)] = \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\varepsilon\frac{\mathrm{d}y}{\mathrm{d}t} + y = 0$$

Problème réduit

$$\mathcal{L}_0[y_0(t)] = \frac{\mathrm{d}^2 y_0}{\mathrm{d}t^2} + y_0 = 0$$

$$\frac{\beta l_0}{mLk} \gg \frac{l_0^2}{mL^2 k} \qquad L = \frac{\beta l_0}{mk} \qquad \varepsilon = \frac{mk}{\beta^2}$$

$$\mathcal{L}_\varepsilon[y(t,\varepsilon)] = \varepsilon\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + y = 0$$

Problème réduit

$$\mathcal{L}_0[y_0(t)] = \frac{\mathrm{d}y_0}{\mathrm{d}t} + y_0 = 0$$

Les problèmes réguliers

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\varepsilon\frac{\mathrm{d}y}{\mathrm{d}t} + y = 0 \qquad \text{avec} \qquad y(0) = 0 \quad \text{et} \quad \frac{\mathrm{d}y}{\mathrm{d}t}(0) = 1$$

$$y(t,\varepsilon) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \dots$$

$$\mathcal{O}(0) \qquad \frac{\mathrm{d}^2 y_0}{\mathrm{d}t^2} + y_0 = 0 \qquad \text{avec } y_0(0) = 0 \quad \text{et} \quad \frac{\mathrm{d}y_0}{\mathrm{d}t}(0) = 1$$

$$\mathcal{O}(1) \qquad \frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} + y_1 = -2\frac{\mathrm{d}y_0}{\mathrm{d}t} \qquad \text{avec } y_1(0) = 0 \quad \text{et} \quad \frac{\mathrm{d}y_1}{\mathrm{d}t}(0) = 0$$

$$y_0 = \sin(t) \qquad \text{et} \qquad y_1 = -t\sin(t)$$

$$y = (1-\varepsilon t)\sin(t) + \dots$$

$$y(t,\varepsilon) = \frac{\mathrm{e}^{-\varepsilon t}}{\sqrt{1-\varepsilon^2}}\sin(t)\sqrt{1-\varepsilon^2}$$

Les problèmes singuliers
Modèle de Friedrichs

$$\mathcal{L}_\varepsilon[y(x,\varepsilon)] = \varepsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - a = 0$$

$$\text{avec} \qquad y(0) = 0 \quad \text{et} \quad y(1) = 1$$

$$\text{et} \qquad 0 < a < 1 \qquad 0 < x < 1$$

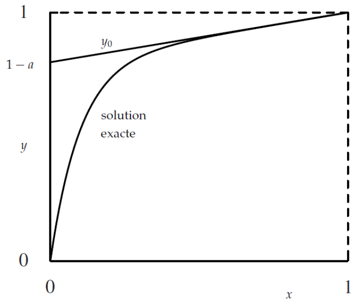
$$\mathcal{L}_0[y_0(x)] = \frac{\mathrm{d}y_0}{\mathrm{d}x} - a = 0$$

$$y_0 = ax + A$$

$$y(x,\varepsilon) = ax + (1-a)\frac{1-\mathrm{e}^{-\frac{x}{\varepsilon}}}{1-\mathrm{e}^{-\frac{1}{\varepsilon}}}$$

$$x > 0, \qquad y = ax + 1 - a + \dots$$

$$\lim_{\varepsilon \to 0} y(x,\varepsilon) = y_0(x) = ax + 1 - a$$



$$X = \frac{x}{\varepsilon}$$

$$y(x,\varepsilon) = a\varepsilon X + (1-a)\frac{1-\mathrm{e}^{-X}}{1-\mathrm{e}^{-\frac{1}{\varepsilon}}}$$

$$\lim_{\varepsilon\to 0} y(x,\varepsilon) = Y_0(X) = (1-a)\left(1-\mathrm{e}^{-X}\right)$$

Remarque étonnante

$$\lim_{X\to\infty} Y_0(X) = \lim_{x\to 0} y_0(x) = 1-a$$

Lagerstrom, Cole, Van Dyke, Eckhaus

Méthode des Développements Asymptotiques Raccordés (MDAR)

$$Y_\alpha(X_\alpha, \varepsilon) \equiv y(x, \varepsilon)$$

$$\varepsilon^{1-2\alpha} \frac{\mathrm{d}^2 Y_\alpha}{\mathrm{d}X_\alpha^2} + \varepsilon^{-\alpha} \frac{\mathrm{d}\, Y_\alpha}{\mathrm{d}X_\alpha} = a \qquad \text{avec} \qquad X_\alpha = \frac{x}{\varepsilon^\alpha}$$

$$a = 1 \qquad X_\alpha = X \qquad Y_\alpha = Y$$

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}X^2} + \frac{\mathrm{d}\, Y}{\mathrm{d}X} = \varepsilon a$$

$$\frac{\mathrm{d}^2 Y_0}{\mathrm{d}X^2} + \frac{\mathrm{d}\, Y_0}{\mathrm{d}X} = 0$$

$$Y_0(X) = A\left(1 - \mathrm{e}^{-X}\right)$$

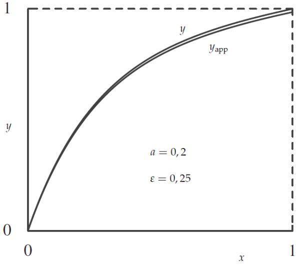
Zone de recouvrement

$$X_\beta = \frac{x}{\varepsilon^\beta} \qquad 0 < \beta < 1$$

$$y_0(x) = 1 - a + a\varepsilon^\beta X_\beta = 1 - a + \dots$$

$$Y_0(X) = A\left[1 - \exp\left(-\frac{X_\beta}{\varepsilon^{1-\beta}}\right)\right] = A + \dots$$

$$y_{app} = y_0(x) + Y_0(X) - (1-a) = ax + (1-a)\left(1 - \mathrm{e}^{-X}\right)$$



$a = 0.2$

$\varepsilon = 0.25$

Méthode des Approximations Successives Complémentaire (MASC)

$$y_{a1} = y_0(x) + Y_0^*(X)$$

$$\mathcal{L}_\varepsilon[y(x,\varepsilon)] = \varepsilon\frac{\mathrm{d}^2 y_0}{\mathrm{d}x^2} + \frac{\mathrm{d}y_0}{\mathrm{d}x} - a + \frac{1}{\varepsilon}\left[\frac{\mathrm{d}^2 Y_0^*}{\mathrm{d}X^2} + \frac{\mathrm{d}\,Y_0^*}{\mathrm{d}X}\right] = \frac{1}{\varepsilon}\left[\frac{\mathrm{d}^2 Y_0^*}{\mathrm{d}X^2} + \frac{\mathrm{d}\,Y_0^*}{\mathrm{d}X}\right]$$

$$Y_0^* X = A + B\mathrm{e}^{-X}$$

$$Y_0^*(0) = a - 1 \text{ et } Y_0^*\left(\frac{1}{\varepsilon}\right) = 0$$

$$Y_0^*(X) = (1-a)\frac{\mathrm{e}^{-\frac{1}{\varepsilon}} - \mathrm{e}^{-X}}{1 - \mathrm{e}^{-\frac{1}{\varepsilon}}}$$

$$Y_0^* X = (a-1)\mathrm{e}^{-X}$$

$$y_{app} = y_{a1} = ax + (1-a)\left(1 - \mathrm{e}^{-X}\right)$$

La Méthode des échelles multiples

$$y(x,\varepsilon) \equiv Y(x,X,\varepsilon)$$

$$\frac{\partial^2 Y}{\partial X^2} + \frac{\partial Y}{\partial X} + \varepsilon\left(2\frac{\partial^2 Y}{\partial X \partial x} + \frac{\partial Y}{\partial x} - a\right) + \varepsilon^2 \frac{\partial^2 Y}{\partial x^2} = 0$$

$$\text{avec} \qquad 0 < x < 1, \qquad 0 < X < \frac{1}{\varepsilon}$$

$$Y(x,X,\varepsilon) = Y_0(x,X) + \varepsilon Y_1(x,X) + \dots$$

$\mathcal{O}(0)$

$$\frac{\partial^2 Y_0}{\partial X^2} + \frac{\partial Y_0}{\partial X} = 0 \qquad Y_0(0,0) = 0 \text{ et } Y_0(1,\infty) = 1$$

$$Y_0(x,X) = A(x) + B(x)\mathrm{e}^{-X}$$

$$A(0) + B(0) = 0 \text{ et } A(1) = 1$$

$\mathcal{O}(1)$

$$\frac{\partial^2 Y_1}{\partial X^2} + \frac{\partial Y_1}{\partial X} = a - \left(2\frac{\partial^2 Y_0}{\partial X \partial x} + \frac{\partial Y_0}{\partial x}\right)$$

$$\frac{\partial^2 Y_1}{\partial X^2} + \frac{\partial Y_1}{\partial X} = a - \frac{\mathrm{d}A}{\mathrm{d}x} + \frac{\mathrm{d}B}{\mathrm{d}x}\mathrm{e}^{-X}$$

$$Y_1(x, X) = C(x) + D(x)\mathrm{e}^{-X} + X\left(a - \frac{\mathrm{d}A}{\mathrm{d}x}\right) - \frac{\mathrm{d}B}{\mathrm{d}x}X\mathrm{e}^{-X}$$

"chaque terme du second ordre ne puisse être plus singulier qu'un terme du premier ordre"

$$a - \frac{\mathrm{d}A}{\mathrm{d}x} = 0 \quad \text{et,} \quad \frac{\mathrm{d}B}{\mathrm{d}x} = 0$$

$$A(x) = ax + 1 - a \quad \text{et,} \quad B(x) = a - 1$$

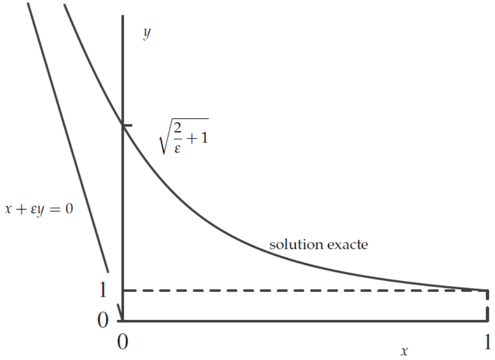
$$Y_0(x, X) = ax + (1 - a)\left(1 - \mathrm{e}^{-X}\right)$$

La Méthode de Poincaré–Lighthill (PLK)

Strained coordinates method dite “méthode PL”

$$\mathcal{L}_\varepsilon y = (x + \varepsilon y)\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$$

avec $y(1) = 1$ pour $0 < x < 1.$



Solution exacte

$$y(x,\varepsilon) = -\frac{x}{\varepsilon} + \sqrt{\frac{x^2}{\varepsilon^2} + \frac{2}{\varepsilon} + 1}$$

$$y(x,\varepsilon) = y_0(s) + \varepsilon y_1(s) + \varepsilon^2 y_2(s) + \dots$$

avec $\qquad x(s,\varepsilon) = s + \varepsilon x_1(s) + \varepsilon^2 x^2(s) + \dots$

$$s\frac{\mathrm{d}y_0}{\mathrm{d}s} + y_0 = 0 \qquad \text{et} \qquad y_0(s=1) = 1$$

$$y_0(s) = \frac{1}{s}$$

$$s\frac{\mathrm{d}y_1}{\mathrm{d}s} + y_1 = \frac{\mathrm{d}y_0}{\mathrm{d}s}\left(s\frac{\mathrm{d}x_1}{\mathrm{d}s} - x_1 - y_0\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(sy_1) = \frac{1}{s^2}\left(x_1 + \frac{1}{s} - s\frac{\mathrm{d}x_1}{\mathrm{d}s}\right)$$

$$y_1(s) = \frac{A}{s} - \frac{1}{s^2}\left[x_1(s) + \frac{1}{2s}\right]$$

"les approximations d'ordre supérieur ne doivent pas être plus singulières que la première approximation"

$$\frac{1}{s}\left[x_1(s) + \frac{1}{2s}\right] = B(s)$$

$$B(s) = A$$

$$x_1(s) = As - \frac{1}{2s}$$

$$x_1(s) = \frac{1}{2}\left(s - \frac{1}{s}\right)$$

$$y(x,\varepsilon) = \frac{1}{s} + \dots$$

$$x(s,\varepsilon) = s + \frac{\varepsilon}{2}\left(s - \frac{1}{s}\right) + \dots$$

La Méthode du groupe de renormalisation

$$\mathcal{L}_\varepsilon[y(t,\varepsilon)] = \frac{\mathrm{d}y}{\mathrm{d}t} + \varepsilon y = 0$$

DA "naïf "

$$y(t,\varepsilon) = A_0[1 - \varepsilon(t - t_0)] + \dots$$

$$A_0 = [1 + \varepsilon a(t_0, \mu)]A(\mu)$$

$$y = A(\mu)[1 + \varepsilon a(t_0, \mu) - \varepsilon(t - \mu) - \varepsilon(\mu - t_0)] + \dots$$

$$a = \mu - t_0$$

$$y(t,\varepsilon) = A(\mu)[1 - \varepsilon(t - \mu)] + \dots$$

critère de renormalisation

$$\frac{\partial y}{\partial \mu} = 0$$

$$\frac{\mathrm{d}A}{\mathrm{d}\mu} + \varepsilon A = 0$$

$$\mu = t$$

$$y(t, \varepsilon) = A_1 \mathrm{e}^{-\varepsilon t} + \dots$$

Les développements asymptotiques

Les fonctions d'ordre.

$$\delta(\varepsilon) \in E \qquad 0 < \varepsilon \leq \varepsilon_0$$

$$\lim_{\varepsilon \to 0} \delta(\varepsilon) \qquad \text{existe au sens large}$$

$$\forall \delta_1 \in E, \forall \delta_2 \in E, \delta_1 \delta_2 \in E$$

$$\varepsilon^n \left[1 + \sin^2(1/\varepsilon)\right] \notin E$$

Notation de Hardy : $\varepsilon \to 0$

$$\delta_1 \preceq \delta_2 \qquad \frac{\delta_1}{\delta_2} \text{ borné}$$

$$\delta_1 \prec \delta_2 \qquad \frac{\delta_1}{\delta_2} \to 0$$

$$\delta_1 \approx \delta_2 \qquad \lim_{\varepsilon \to 0} \frac{\delta_1}{\delta_2} = \lambda$$

$$\delta_1 \cong \delta_2 \qquad \lambda = 1$$

$$\varepsilon^2 \preceq \varepsilon \qquad \varepsilon^2 \prec \varepsilon \qquad \varepsilon \preceq \frac{\varepsilon}{1+\varepsilon} \qquad \varepsilon \approx \frac{\varepsilon}{1+\varepsilon} \qquad \varepsilon \cong \frac{\varepsilon}{1+\varepsilon}$$

$$R(\delta_1, \delta_2) : \delta_1 \approx \delta_2 \quad \text{ou} \quad \delta_1 \preceq \delta_2$$

R réflexive, transitive et antisymétrique : ordre total

Contre-exemple :

$$\delta_1 = \varepsilon \qquad \delta_2^* = \varepsilon \left[1 + \sin^2(1/\varepsilon)\right]$$

$$\delta_1 \preceq \delta_2^* \qquad \delta_2^* \preceq \delta_1 \qquad \delta_1 \neq \delta_2^*$$

Notation de Landau : $\varphi(x, \varepsilon); D \times (0 < \varepsilon \leq \varepsilon_0)$

$$\varphi = \mathcal{O}(\delta) \qquad \exists K : \|\varphi\| \leq K\delta$$

$$\varphi = o(\delta) \qquad \lim_{\varepsilon \to 0} \frac{\|\varphi\|}{\delta} = 0$$

$$\varphi = \mathcal{O}_S(\delta) \qquad \exists K : \lim_{\varepsilon \to 0} \frac{\|\varphi\|}{\delta} = K$$

$$\sin\left(\frac{1}{\varepsilon}\right) = \mathcal{O}(1)$$

$$\|\varphi\| = \mathrm{Max}_{\mathcal{D}}|\varphi|$$

$$\varphi = \exp\left(-\frac{x}{\varepsilon}\right) \qquad D = [0, 1]$$

$$\|\varphi\| = \mathcal{O}_S(1)$$

$$\|\varphi\| = \sqrt{\int_{\mathcal{D}} \varphi^2 \mathrm{d}x}$$

$$\varphi = \mathcal{O}_S(\sqrt{\varepsilon})$$

Développements Asymptotiques (DA)

DA généralisé

$$\varphi(x,\varepsilon) = \sum_{n=1}^{m} \delta_n(\varepsilon)\varphi_n(x,\varepsilon) + \mathcal{O}[\delta_m(\varepsilon)]$$

δ_n suite asymptotique

$\forall n, \quad \delta_{n+1} \prec \delta_n$ et où $\varphi_n = \mathcal{O}_S(1)$

DA régulier

$$\varphi(x,\varepsilon) = \sum_{n=1}^{m} \delta_n(\varepsilon)\varphi_n(x) + \mathcal{O}[\delta_m(\varepsilon)]$$

$$E^{(m)}\varphi = \sum_{n=1}^{m} \delta_n(\varepsilon)\varphi_n(x)$$

$$\varphi(x,\varepsilon) = E^{(m)}\varphi + \mathcal{O}[\delta_m(\varepsilon)]$$

Processus constructif

$$\forall h < m$$

$$\lim_{\varepsilon\to 0} \frac{\varphi(x,\varepsilon) - \sum_{i=1}^{h} \delta_i(\varepsilon)\varphi_i(x)}{\delta_{h+1}} = \varphi_{h+1}(x)$$

$$\varphi(x,\varepsilon) = \left(1 - \frac{\varepsilon}{1+\varepsilon}x\right)^{-1}$$

$$\varphi(x,\varepsilon) = 1 + \sum_{n=1}^{m} \delta_n(\varepsilon)x^n + o[\delta_m(\varepsilon)]$$

Avec $\delta_n(\varepsilon) = \dfrac{\varepsilon}{1+\varepsilon} \Longrightarrow \varphi(x,\varepsilon) = 1 + \sum_{n=1}^{m} \varepsilon^n x(x-1)^{n-1} + o[\varepsilon^m]$

$$\sin(2\varepsilon) = 2\varepsilon - \frac{4}{3}\varepsilon^3 + \frac{4}{15}\varepsilon^5 + \mathcal{O}(\varepsilon^7)$$

$$\sin(2\varepsilon) = 2\tan(\varepsilon) - 2\tan^3(\varepsilon) - 2\tan^5(\varepsilon) + \mathcal{O}(\varepsilon^7)$$

$$\sin(2\varepsilon) = 6\frac{\varepsilon}{3+2\varepsilon^2} - \frac{756}{5}\left(\frac{\varepsilon}{3+2\varepsilon^2}\right)^5 + \mathcal{O}(\varepsilon^7)$$

$$\bar{\varepsilon} = \frac{\varepsilon}{3+2\varepsilon^2}$$

Développement de Taylor

$$\varphi(\varepsilon) = \varphi(0) + \varepsilon\varphi'(0) + \ldots + \varepsilon^m \frac{\varphi^m(0)}{m!} + \mathcal{O}(\varepsilon^m)$$

Séries divergentes : Hardy 1949

$$S = 1 + 2 + 4 + 8 + \ldots + 2^m + \ldots$$

$$2S = S - 1 \qquad S = -1$$

Génie ou ignorance ? Euler

$$f(x) = \frac{1}{1-x}$$

$$f(x) = 1 + x + x^2 + \ldots + x^m + \ldots$$

Série asymptotique convergente

$$e^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^m}{m!} + \ldots$$

convergente $\forall x$

DA voisinage de $x = 0$

$$f(\varepsilon) = \text{Erfc}\left(\frac{1}{\varepsilon}\right) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{\varepsilon}} e^{-t^2} dt$$

$$\text{Erfc}\left(\frac{1}{\varepsilon}\right) = 1 - \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!\varepsilon^{2n+1}}$$

$$\text{Erfc}\left(\frac{1}{\varepsilon}\right) = \frac{\varepsilon}{\sqrt{\pi}} e^{-\frac{1}{\varepsilon^2}} \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \times 3 \times \ldots \times (2n-1)}{2^n} \varepsilon^{2n}\right]$$

Premier terme $\quad \varepsilon = \frac{1}{100} \quad$ précision $\mathcal{O}(10^{-8})$

Attention
Fonction modèle

$$f(x,\varepsilon) = \sqrt{x+\varepsilon}$$

$$f(x,\varepsilon) = \sqrt{x} + \mathcal{O}(1)$$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{1}{2\sqrt{x+\varepsilon}}$$

$$\frac{\mathrm{d}f_0}{\mathrm{d}x} = \frac{1}{2\sqrt{x}} \quad \text{avec } f_0 = \sqrt{x}$$

MDAR et MASC

Méthode des Développements Asymptotiques Raccordés : MDAR

Technique du raccord intermédiaire

Principe de Van Dyke : PVD

Principe Modifié de Van Dyke : PMVD

Approximation Uniformément Valable : AUV

L'opérateur d'expansion

$\phi(x, \varepsilon)$ dans $\mathcal{D}$: $[0, 1]$

$$\phi(x, \varepsilon) = \sum_{i=1}^{n} \delta_0^{(i)}(\varepsilon)\phi_0^{(i)}(x) + \mathcal{O}\left(\delta_0^{(n)}\right)$$

Opérateur d'expansion $\quad E_0^{(n)}$

$$E_0^{(n)}(\phi) = \sum_{i=1}^{n} \delta_0^{(i)}(\varepsilon)\phi_0^{(i)}(x)$$

$$\phi(x, \varepsilon) - E_0^{(n)}\phi = \mathcal{O}\left(\delta_0^{(n)}\right)$$

DA de ϕ pour $m < n$

$$E_0^{(m)} E_0^{(n)}\phi = E_0^{(m)}\phi + \mathcal{O}\left(\delta_0^{(m)}\right)$$

Fonctions de jauge

$$E_0^{(m)} E_0^{(n)} \phi = E_0^{(m)} \phi$$

ϕ singulière

$$\mathcal{D}_0 : 0 < A_0 \leq x \leq 1$$

$$\exp\left(-\frac{x}{\varepsilon}\right)$$

Développement extérieur
domaine extérieur $\mathcal{D}_0$
x : variable extérieure.

Un exercice préliminaire

$$\phi(x,\varepsilon) = \frac{2}{\sqrt{1-4\varepsilon}} \exp\left(-\frac{x}{2\varepsilon}\right) \operatorname{sh}\left(\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x\right)$$

$$\phi(x,\varepsilon) = E_0^{(2)}\phi + \mathcal{O}(\varepsilon)$$

$$E_0^{(2)}\phi = \mathrm{e}^{-x} + \varepsilon \mathrm{e}^{-x}(2-x)$$

Processus limite extérieur

“x fixé, $\varepsilon \to 0$” Perturbation singulière $\phi(0,\varepsilon) = 0$

$$E_0^{(1)}\phi = \mathrm{e}^{-x}$$

Variable de couche limite
Variable intérieure

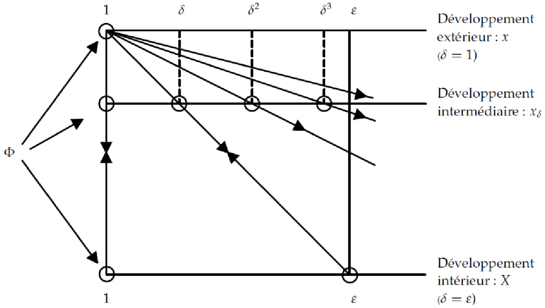
$$X = \frac{x}{\varepsilon}$$

Processus limite intérieur

"X fixé, $\varepsilon \to 0$"

$$\phi(x,\varepsilon) = E_1^{(2)}\phi + \mathcal{O}(\varepsilon)$$

$$E_1^{(2)}\phi = 1 - \mathrm{e}^{-X} + \varepsilon\left[(2-X) - (2+X)\mathrm{e}^{-X}\right]$$



$$1 + \varepsilon(2 - X) = E_1^{(2)}E_0^{(2)}\phi \equiv E_0^{(2)}E_1^{(2)}\phi = 1 - x + 2\varepsilon$$

Variables locales

$$x_\nu = \frac{x}{\delta_\nu(\varepsilon)}$$

$$0 < \nu_1 < \nu_2 \qquad \delta_{\nu_2} \prec \delta_{\nu_1}$$

$$\mathcal{D}_\nu : A_\nu \leq x_\nu \leq B_\nu$$

$$\phi(x,\varepsilon) = \sum_{i=1}^{n_\nu} \delta_\nu^{(i)}(\varepsilon)\phi_\nu^{(i)}(x_\nu) + \mathcal{O}\left(\delta_\nu^{(n_\nu)}\right)$$

$$E_\nu^{(n_\nu)}\phi = \sum_{i=1}^{n_\nu} \delta_\nu^{(i)}\varepsilon\phi_\nu^{(i)}(x_\nu)$$

$$\phi(x,\varepsilon) - E_\nu^{(n_\nu)}\phi = \mathcal{O}\left(\delta_\nu^{(n_\nu)}\right)$$

Approximation de ϕ à l'ordre δ

$$\phi - E_\nu \phi = \mathcal{O}(\delta)$$

Fonction régulière

$$E_\nu E_0 \phi = E_\nu \phi$$

$E_0\phi$ contient $E_\nu\phi$
$E_\mu\phi$ contient $E_\nu\phi$

$$E_\nu E_\mu \phi = E_\nu \phi$$

$\nu = 1$ significatif

$E_1\phi$ n'est pas contenu dans $E_0\phi$

On aimerait bien

$$E_\nu E_0 \phi = E_\nu \phi$$

$$\text{et, } E_\nu E_1 \phi = E_\nu \phi$$

$$\text{et, } E_\nu E_0 \phi = E_\nu E_1 \phi$$

Théorème de Kaplun
théorème d'extension

$$\forall E_\mu \phi \text{ de } \phi \; \exists \nu : E_\nu E_\mu \phi = E_\nu \phi$$

Multicouche

$$\phi(x,\varepsilon) = 1 + x + a_1 \mathrm{e}^{-\frac{x}{\sqrt{\varepsilon}}} + a_2 \mathrm{e}^{-\frac{x}{\varepsilon}} + a_3 \mathrm{e}^{-\frac{x}{\varepsilon^2}}$$

$$x_1 = \frac{x}{\sqrt{\varepsilon}} \qquad x_2 = \frac{x}{\varepsilon} \qquad x_3 = \frac{x}{\varepsilon^2}$$

$$E_0^{(1)}\phi = 1 + x$$

$$E_1^{(1)}\phi = 1 + a_1 \mathrm{e}^{-x_1}$$

$$E_2^{(1)}\phi = 1 + a_1 + a_2 \mathrm{e}^{-x_2}$$

$$E_2^{(1)}\phi = 1 + a_1 + a_2 + a_3 \mathrm{e}^{-x_2}$$

Une seule couche $\phi(x,\varepsilon) = 1 + x + \mathrm{e}^{-\frac{x}{\varepsilon}}$

$$\delta_\nu(\varepsilon) = \varepsilon^\nu$$

Ordre 1

$$E_0\phi = 1 + x \text{ pour } \nu = 0$$

$$E_\nu\phi = 1 \text{ pour } 0 < \nu < 1$$

$$E_1\phi = 1 + \mathrm{e}^{-x_1} \text{ pour } \nu = 1$$

$$\mu < 1 \qquad 1 = E_\mu E_0\phi = E_\mu\phi = 1$$

$E_0\phi$ contient $E_\mu\phi$

$$\mu < 1 \qquad 1 = E_\mu E_1\phi = E_\mu\phi = 1$$

$E_1\phi$ contient $E_\mu\phi$

$$0 < \nu < 1, \;\; E_\nu E_0\phi = E_\nu E_1\phi = 1$$

Hélas, ce serait trop simple !

Un logarithme

Propriété contrariante : $\forall m,\ \ln(\varepsilon^m) = m\ln(\varepsilon)$

$$\phi(x,\varepsilon) = \frac{1}{\ln(x)} + \frac{1}{\ln(\varepsilon)}\mathrm{e}^{-\frac{x}{\varepsilon}}$$

Ordre $-\frac{1}{\ln(\varepsilon)}$

$$E_0\phi = \frac{1}{\ln(x)} \qquad \text{pour } \nu = 0$$

$$E_\nu\phi = \frac{1}{\nu\ln(\varepsilon)} \qquad \text{pour } 0 < \nu < 1$$

$$E_1\phi = \frac{1+\mathrm{e}^{-x_1}}{\ln(\varepsilon)} \qquad \text{pour } \nu = 1$$

$$E_\nu E_1\phi = \frac{1}{\ln(\varepsilon)} \neq \frac{1}{\nu\ln(\varepsilon)} = E_\nu\phi$$

$$0 < \nu < 1; \qquad E_\nu E_0\phi \neq E_\nu E_1\phi$$

Miracle : $$E_1 E_0\phi = E_0 E_1\phi = \frac{1}{ln(\varepsilon)}$$

Le Principe de Van Dyke (PVD)

1964 : Van Dyke

$$nm = mn$$

$$E_1^{(m)} E_0^{(n)} \phi = E_0^{(n)} E_1^{(m)} \phi$$

$$\phi_{\text{ap}} = E_0^{(n)} \phi + E_1^{(m)} \phi - E_1^{(m)} E_0^{(n)} \phi$$

Exemple préliminaire

$$E_0^{(2)} \phi = \mathrm{e}^{-x} + \varepsilon \mathrm{e}^{-x} (2 - x)$$

$$E_1^{(2)} \phi = \left(1 - \mathrm{e}^{-X}\right) + \varepsilon \left[(2 - X) - (2 + X)\mathrm{e}^{-X}\right]$$

$$E_1^{(1)} E_0^{(1)} \phi = E_0^{(1)} E_1^{(1)} \phi = 1$$

$$\phi_{\text{ap}} = \mathrm{e}^{-x} - \mathrm{e}^{-X}$$

$$E_1^{(1)} E_0^{(2)} \phi = E_0^{(2)} E_1^{(1)} \phi = 1$$
$$\phi_{\text{ap}} = \mathrm{e}^{-x} - \mathrm{e}^{-X} + \varepsilon \mathrm{e}^{-x}(2 - x)$$

$$E_1^{(2)} E_0^{(1)} \phi = E_0^{(1)} E_1^{(2)} \phi = 1 - \varepsilon X$$
$$\phi_{\text{ap}} = \mathrm{e}^{-x} - \mathrm{e}^{-X} + \varepsilon \left[(2 - (2 + X)\mathrm{e}^{-X} \right]$$

$$E_1^{(2)} E_0^{(2)} \phi = E_0^{(2)} E_1^{(2)} \phi = 1 + \varepsilon(2 - X)$$
$$\phi_{\text{ap}} = \mathrm{e}^{-x} - \mathrm{e}^{-X} + \varepsilon \left[(2 - x)\mathrm{e}^{-x} - (2 + X)\mathrm{e}^{-X} \right]$$

Le Principe Modifié de Van Dyke (PMVD)

Ordre donné δ

$$E_1 E_0 \phi = E_0 E_1 \phi$$

$$\phi_{\text{ap}} = E_0 \phi + E_1 \phi - E_1 E_0 \phi$$

Exemple précédant

$$\phi(x,\varepsilon) = \frac{1}{\ln x} + \frac{1}{\ln \varepsilon} \mathrm{e}^{-\frac{x}{\varepsilon}}$$

$$\mathcal{O}\left(\frac{1}{(\ln \varepsilon)^2}\right) \qquad \text{Suite : } 1,\ -\frac{1}{\ln \varepsilon},\ \frac{1}{(\ln \varepsilon)^2}$$

$$E_0\phi = \frac{1}{\ln x}$$

$$E_1\phi = \frac{1 + \mathrm{e}^{-x_1}}{\ln \varepsilon} - \frac{\ln x_1}{(\ln \varepsilon)^2}$$

PVD $$\frac{1}{\ln \varepsilon} - \frac{\ln x_1}{(\ln \varepsilon)^2} = E_1^{(2)} E_0^{(1)} \phi \neq E_0^{(1)} E_1^{(2)} \phi = \frac{2}{\ln \varepsilon}$$

PMVD $$\frac{1}{\ln \varepsilon} - \frac{\ln x_1}{(\ln \varepsilon)^2} = E_1 E_0 \phi = E_0 E_1 \phi = \frac{2}{\ln \varepsilon} - \frac{\ln x}{(\ln \varepsilon)^2}$$

$$\phi_{\mathrm{ap}} = \phi$$

Réflexions sur le raccord asymptotique

$$\phi(x,\varepsilon) = 1 + \mathrm{e}^{-\frac{x}{\varepsilon}} + \varepsilon \ln \frac{x}{\varepsilon}$$

$$E_0\phi = 1 - \varepsilon \ln \varepsilon + \varepsilon \ln x$$

$$E_1\phi = 1 + \mathrm{e}^{-x_1} + \varepsilon \ln x_1$$

$$\mathcal{O}(-\varepsilon \ln \varepsilon) \text{ : Pas de raccord}$$

$$1 - \varepsilon \ln \varepsilon = E_1 E_0 \phi \neq E_0 E_1 \phi = 1$$

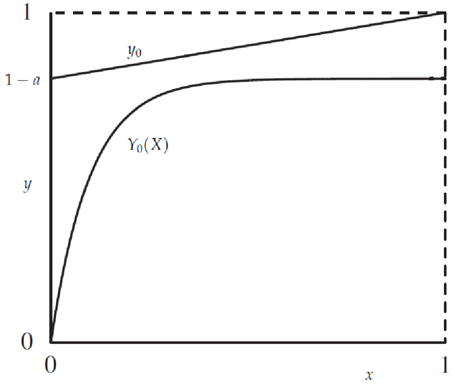
Van Dyke :

"Fortunately, since the two expansions have a common region of validity, it is easy to construct from them a single uniformly valid expansion"

Modèle de Friedrichs

$$E_0 y = y_0(x) = ax + 1 - a$$

$$E_1 y = Y_0(X) = (1 - a)\left(1 - \mathrm{e}^{-X}\right)$$



Méthode déficitaire

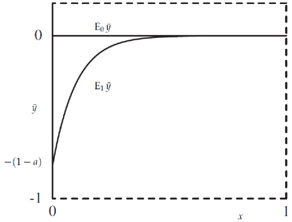
Couche limite corrective MAUV

$$\tilde{y} = y - y_0$$

$$\mathcal{O}(1) \qquad E_0\tilde{y} = 0$$

$$E_1\tilde{y} = E_1(y - E_0 y)$$

$$E_1\tilde{y} = -(1-a)\mathrm{e}^{-X}$$



AUV de $\tilde{y}$: $\tilde{y} = E_1\tilde{y} + \mathcal{O}(1)$

Méthode des Approximations successives complémentaires.
On doit supposer la structure d'une AUV et en déduire la méthode permettant de la construire.

$$\phi_a(x, X, \varepsilon) = \sum_{i=1}^{n} \bar{\delta}_i(\varepsilon) \left[\bar{\varphi}_i(x, \varepsilon) + \psi_i(X, \varepsilon)\right]$$

$$\phi = \phi_a + o(\bar{\delta}_n)$$

$$\text{AUV} : \phi_a = \phi_{ar} + o(\delta_m)$$

$$\bar{\delta}_n = \mathcal{O}(\delta_m)$$

$$\phi_{ar}(x, X, \varepsilon) = \sum_{i=1}^{m} \delta_i(\varepsilon)[\varphi_i(x) + \psi_i(X)]$$

$$E_1 E_0 \phi = E_0 E_1 \phi$$

$$\phi_{ap} = E_0 \phi + E_1 \phi - E_1 E_0 \phi$$

MASC recommandée

Structure complexe

$$\phi(x,\varepsilon) = 1 + \frac{\varepsilon^2}{x+\varepsilon^2}\mathrm{e}^{-\frac{x}{\varepsilon}}$$

Ordre de grandeur ?

$$L_\varepsilon[\phi(x,\varepsilon)] = (x+\varepsilon)\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \frac{\mathrm{d}\phi}{\mathrm{d}x} - 1 = 0$$

$$\phi(x=0) = 0 \qquad \phi(x=1) = 2$$

$$L_0\varphi_1 = x\frac{\mathrm{d}^2\varphi_1}{\mathrm{d}x^2} + \frac{\mathrm{d}\varphi_1}{\mathrm{d}x} - 1 = 0$$

$$\phi_1 = 1 + x + A_1 \ln(x)$$

$$\phi_a = \varphi_1 + \psi_1(X,\varepsilon)$$

$$L_\varepsilon \phi_a = -\varepsilon \frac{A_1}{x^2} + \frac{1}{\varepsilon}\left[(1+X)\frac{\mathrm{d}^2\psi_1}{\mathrm{d}X^2} + \frac{\mathrm{d}\psi_1}{\mathrm{d}X}\right]$$

$$A_1 = 0 \qquad \text{et,} \quad (1+X)\frac{\mathrm{d}^2\psi_1}{\mathrm{d}X^2} + \frac{\mathrm{d}\psi_1}{\mathrm{d}X} = 0$$

$$\psi_1 = B_1 \ln(1+X) + B_2$$

$$B_1 = \frac{1}{\ln\left(1+\frac{1}{\varepsilon}\right)} \qquad B_2 = -1$$

$$\phi = \phi_a = x + \frac{\ln\left(1+\frac{x}{\varepsilon}\right)}{\ln\left(1+\frac{1}{\varepsilon}\right)}$$

$$L_\varepsilon \phi_a = R(x, X, \varepsilon)$$

R petit en un certain sens : $\|\phi - \phi_a\|$ petit