Abstract

The purpose of this article is to develop an asymptotic Eulerian approach in plate and shell theory valid for large displacements. With this Eulerian approach, the dead load assumption generally used with Lagrangian approaches can be dropped. We also obtain a membrane model which takes into account following forces, whose direction can vary during large displacements.

Keywords: Elastic shell theory; Non-linear membrane model; Asymptotic methods; Eulerian approach; Large displacements

1. Introduction

In classical approaches of plate and shell theory which use a Lagrangian description of the initial three-dimensional problem [1–8,10] the forces acting on the structure are generally assumed to be dead. 1 This assumption is reasonable within the framework of small displacements, where the initial and the deformed configurations are close together. However, for large displacements, the forces acting in the deformed configuration cannot be considered as dead and this assumption must be dropped. To take into account the real physical forces acting in the deformed configuration, we propose to develop an asymptotic Eulerian approach in plate and shell theory. Eulerian shell formulations turn out to be very effective for fluid–structure coupling problems [12].

The idea is to apply the asymptotic approach already developed by the authors for plates [13–16] and shells [17,18] in the framework of an Eulerian formulation of the three-dimensional problem [19]. First, for Saint-Venant Kirchhoff material subjected to small strains, we deduce a linear constitutive law which links the Cauchy stress tensor \( \sigma^* \) to the Almansi–Euler strain tensor \( A^* \). This enables us to write the three-dimensional Eulerian equilibrium equations in a simple form.

Then using an intrinsic formalism, the equilibrium equations are decomposed onto the tangent plane and the normal to the middle surface of the shell in its deformed configuration. Introducing reference quantities of the problem, we write the equilibrium equations in a dimensionless form. This naturally makes appear dimensionless numbers \( \varepsilon \) and \( \mathcal{C} \) characterizing the geometry of the undeformed and deformed shell, and \( (\mathcal{F}_n, \mathcal{F}_t, \mathcal{G}_n, \mathcal{G}_t) \) characterizing the Eulerian forces acting in the deformed configuration.
To apply the classical asymptotic method, the problem is reduced to a one-scale problem where the initial relative thickness $\varepsilon = h_0/L_0$ is chosen as the reference perturbation parameter. The other dimensionless numbers must be linked to $\varepsilon$. First, within the framework of large displacements, we allow the shell to be strongly bent in its deformed configuration. This fixes a priori $\varepsilon = \varepsilon$. On the other hand, to obtain large displacements, we consider that the shell is subjected to a severe level of surface forces $\mathcal{G}_n = \mathcal{G}_i = \varepsilon$ acting in the deformed configuration.

Once reduced to a one-scale problem, the asymptotic expansion of equations leads to a two-dimensional membrane model naturally expressed in the deformed configuration. The Souriau-Valid intrinsic formalism used $[20–23]$ enables to pull back the obtained two-dimensional Eulerian model in the initial configuration by taking into account the direction of the forces. The pulled back equations are then compared to those obtained from a Lagrangian approach $[16,24,25]$.

Finally, in the last sections, we come back to the initial physical variables to compare the Eulerian dimensional membrane equations to those obtained with a direct surfacic approach $[26–28]$.

2. The three-dimensional problem

In what follows we index by a star (*) all dimensional variables. Greek indices take their values in $\{1, 2\}$ and Latin indices take their values in $\{1, 2, 3\}$. On the other hand, within the framework of large displacements, the reference and the current configuration cannot be confused. So the reference configuration variables will be indexed by $(0)$.

Let $\omega^*_{0}$ be a connected surface embedded in $\mathbb{R}^3$, whose diameter is $L_0$, with a “smooth enough” boundary $\gamma^*_{00}$. We note $N_0$ the unit normal to $\omega^*_{0}$ and $C^*_{0}$ its curvature operator.

Let $\gamma^*_{00}$ be the identity mapping on $\omega^*_{0}$, $T\omega^*_{0}$ the tangent bundle of $\omega^*_{0}$ (the disjoint collection of all tangent spaces corresponding to all points $p^*_{0}$ of $\omega^*_{0}$), $I_{T\omega^*_{0}}$ or $I_{\gamma^*_{00}T\omega^*_{0}}$ the identity on $T\omega^*_{0}$, $H^*_{0}$ the orthogonal projection onto $T\omega^*_{0}$ and $J^*_{0}$ the $\pi/2$ rotation around $N_0$.

Let us consider a shell of $2h_0$ thickness, whose middle surface is $\omega^*_{0}$. The shell itself occupies the domain $\mathcal{O}^*_{0}$ in its reference configuration where $\Omega^* = \omega^*_{0} \times ]-h_0, h_0[$ is an open set of $\mathbb{R}^3$. We note $\gamma^*_{0}$ the generic point of $\mathcal{O}^*_{0}$ and $\Gamma^*_{0 \pm} = \omega^*_{0} \times \{-h_0, h_0\}$ the upper and lower faces of the shell. The shell is assumed to be clamped on a portion $\Gamma^*_{01} = \gamma^*_{01} \times [-h_0, h_0]$ of the lateral surface $\Gamma^*_{0} = \gamma^*_{0} \times [-h_0, h_0]$ and free on the other portion $\Gamma^*_{02} = \gamma^*_{02} \times [-h_0, h_0]$, where $(\gamma^*_{01}, \gamma^*_{02})$ is a partition of $\gamma^*_{0}$. Moreover, we consider only thin shells where $h_0 \ll L_0$ and $h_0\|C^*_{0}\|_{\infty} \ll 1$.

We assume that the shell is in equilibrium in its deformed configuration $\mathcal{O}^*_{0}$ under the body forces $f^* : \mathcal{O}^*_{0} \rightarrow \mathbb{R}^3$ and surface force $g^{*\pm} : \Gamma^*_{\pm} \rightarrow \mathbb{R}^3$, where $\Gamma^*_\pm$ denote the upper and lower faces of the deformed configuration. We note $\Gamma^* = \Gamma^*_1 \cup \Gamma^*_2$ the lateral surface of $\mathcal{O}^*_{0}$ where $\Gamma^*_1$ coincides with $\Gamma^*_{01}$ because the shell is clamped on the portion $\Gamma^*_{01}$ of the lateral surface. Contrary to the classical Lagrangian approaches, we consider here the “real physical” Eulerian forces which are defined in the deformed configuration.

The general Eulerian equilibrium equations can be written

$$\text{Div}^* \sigma^* + f^* = 0 \quad \text{in } \Omega^*,$$
$$\sigma^* n^*_\pm = g^{*\pm} \quad \text{on } \Gamma^*_\pm,$$
$$\sigma^* n^* = 0 \quad \text{on } \Gamma^*_2,$$

where $\sigma^*$ denotes the Cauchy stress tensor, $n^*_\pm$ (respectively $n^*$) the unit external normal to the lower and upper faces $\Gamma^*_\pm$ (respectively to the lateral surface $\Gamma^*$). In what follows $\text{Div}^*$ will denote the divergence in the three-dimensional space whereas $\text{div}^*$ will denote the two-dimensional divergence on the middle surface. The divergence of endomorphisms is considered as a linear form.

The three complementary equations will be given by a constitutive law between the Cauchy stress tensor $\sigma^*$ and the Almansi–Euler strain tensor $A^*$. This constitutive law will be deduced from the classical Saint-Venant constitutive law.

2.1. Eulerian constitutive law

In the classical Lagrangian description, we usually consider the mapping

$$\psi^* : \Omega^*_{0} \rightarrow \Omega^*,$$
$$q^*_{0} \mapsto q^* = \psi^*(q^*_{0})$$
whose linear tangent mapping is \( F^* = \partial q^*/\partial q_0 \), where \( q^* \) denotes a generic point of \( \Omega^* \).

In linear elasticity (linear behaviour), the constitutive laws used are generally Lagrangian: Hooke, Saint-Venant Kirchhoff materials. However in the case of small strains, we have the following Proposition:

**Proposition 1.** For small strains, the Saint-Venant Kirchhoff constitutive law \( \Sigma^* = \lambda \text{Tr} E^* I + 2\mu E^* \) implies the following constitutive relation between \( \sigma^* \) and \( A^* \):

\[
\sigma^* = \lambda (\text{Tr} A^*) I + 2\mu A^*,
\]

where \( I \) denotes the identity of \( \mathbb{R}^3 \), \( \lambda \) and \( \mu \) the Lamé constants of the material.

**Proof.** Let us consider the Saint-Venant Kirchhoff constitutive law \( \Sigma^* = \lambda \text{Tr} E^* I + 2\mu E^* \) between the second Piola–Kirchhoff stress tensor \( \Sigma^* \) and the Green–Lagrange strain tensor \( E^* = \frac{1}{2}(F^*F^* - I) \). Let us now express \( \sigma^* \) with respect to \( A^* \). As \( \Sigma^* \) is linked to the Cauchy stress tensor \( \sigma^* \) by the relation \( \sigma^* = (1/\det F^* )F^* \Sigma^* F^* \), we get

\[
\sigma^* = \frac{1}{\det F^*}[\lambda \text{Tr} E^* I + 2\mu E^*] F^*.
\]

Taking into account the definition of the Almansi–Euler strain tensor \( A^* = \frac{1}{2}(F^*F^* - I) \), we have

\[
F^* A^* F^* = \frac{1}{2}(F^*F^* - I) = E^*.
\]

Then we get

\[
\sigma^* = \frac{1}{\det F^*}[\lambda \text{Tr}(F^* F^* A^*) I + 2\mu F^* A^* F^*] F^*
= \frac{1}{\sqrt{\det F^* F^*}} [\lambda \text{Tr}(F^* F^* A^*) F^* F^*]
+ 2\mu(F^* F^* A^*)^T(F^* F^*).
\]

The invertibility of \( F^* \) implies the invertibility of \( F^* F^* - I \) and we have

\[
F^* F^* = (I - 2A^*)^{-1}.
\]

In the case of small strain where \( \|A^*\| \leq 1 \), an expansion at the first order with respect to \( A^* \) leads to

\[
F^* F^* = (I - 2A^*)^{-1} = I + 2A^* + O(A^2)
\]

and relation (3) reduces to

\[
\sigma^* = \lambda (\text{Tr} A^*) I + 2\mu A^* + O(A^2).
\]

Therefore, for small strains, it is equivalent to consider the Saint-Venant Kirchhoff or the Eulerian equivalent constitutive law

\[
\sigma^* = \lambda (\text{Tr} A^*) I + 2\mu A^*.
\]

\( \square \)

### 3. Equilibrium equations

In an Eulerian description, it is more convenient to consider the inverse mapping

\[
\phi^* : \Omega^* \rightarrow \Omega^*_0,
q^* \mapsto q^*_0 = \psi^{-1}(q^*) = \phi^*(q^*).
\]

So we have \( A^* = \frac{1}{2}(I - (\partial \phi^*/\partial q^*) \partial \phi^*/\partial q^*) \) and \( \sigma^* = \lambda \text{Tr} A^* I + 2\mu A^* \). Thus the equilibrium equations can be written as

\[
\begin{align*}
\lambda \frac{\partial \text{Tr} A^*}{\partial q^*} + 2\mu \text{Div} A^* &= -f^* \quad \text{in } \Omega^*, \\
[\lambda \text{Tr} A^* I + 2\mu A^*]n^*_{\pm} &= g^*_{\pm} \quad \text{on } \Gamma^*_\pm, \\
\phi^* &= I \quad \text{on } \Gamma^*_1, \\
[\lambda \text{Tr} A^* I + 2\mu A^*]n^* &= 0 \quad \text{on } \Gamma^*_2.
\end{align*}
\]

#### 3.1. Decomposition of equilibrium equations

In Lagrangian descriptions, we generally consider shells whose thickness is constant and whose middle surface is equidistant to the lower and the upper faces. With an Eulerian approach the problem is more complex because the deformed configuration is an unknown of the problem and the thickness is not a priori constant. However, it is possible to define a “middle surface” noted \( \omega^* \) but which is not necessary equidistant to the lower and the upper faces \( \Gamma^*_\pm \). In what follows, we will just assume that the shell is a sufficiently thin neighbourhood of the middle surface \( \omega^* \) for the decomposition (5) to be valid. Let us denote \( N^* \) the unit normal to a generic point \( p^* \) of the middle surface \( \omega^* \), \( C^* \) its curvature operator, and \( \gamma^* = \gamma^*_1 \cup \gamma^*_2 \) its lateral boundary with \( \gamma^*_1 = \gamma^*_0 \) and \( \gamma^*_2 = \psi(\gamma^*_0) \).

Then a generic point \( q^* \in \Omega^* \) of the shell in its deformed configuration can be decomposed as follows:

\[
q^* = p^* + z^* N^*,
\]

\[ -h^*_-(p^*) \leq z^* \leq h^*_+(p^*), \quad (5) \]
where $p^*$ denotes the orthogonal projection of $q^*$ on the tangent plane $T_{p^*}\omega^*$. We denote $h^*(p^*) = \frac{1}{2}(h^*(p^*) + h^*(p^*))$ the “half-thickness” of the shell in its deformed configuration which is not a priori constant and depends on $p^*$. We assume that the functions $h^*(p^*)$ and $h^*(p^*)$ are smooth enough for the following calculations to be possible.

Now let us split the Eulerian equilibrium equations onto the middle surface and the normal to the deformed middle surface. First, we can decompose easily the gradient $\partial \phi^*/\partial q^*$ onto $T_{p^*}\omega^* \oplus \mathbb{R}N^*$ as

$$
\frac{\partial \phi^*}{\partial q^*} = \frac{\partial \phi^*}{\partial p^*} + \kappa^{-1} \Pi^* \frac{\partial \phi^*}{\partial z^*} + N^*.
$$

Thus, we obtain

$$
\frac{\partial \phi^*}{\partial q^*} = \frac{\partial \phi^*}{\partial p^*} + \kappa^{-1} \Pi^* \frac{\partial \phi^*}{\partial p^*} + \kappa^{-1} \Pi^* \frac{\partial \phi^*}{\partial z^*} + N^*.
$$

where $\Pi^*$ denotes the orthogonal projection on $T\omega^*$ and $\kappa^*$ the endomorphism of the tangent plane defined as follows:

$$
\kappa^* = I_{T\omega^*} - z^* C^*.
$$

To simplify the notations, we can use the equivalent matricial notation in $T_{p^*}\omega^* \oplus \mathbb{R}N^*$:

$$
A^* = \begin{pmatrix} A^*_t & A^*_s & A^*_n \\
\hat{A}^*_t & \hat{A}^*_s & \hat{A}^*_n \\
\end{pmatrix},
$$

with

$$
A^*_t = \frac{1}{2} \left( I_{T_{p^*}\omega^*} - \kappa^{-1} \frac{\partial \phi^*}{\partial p^*} \frac{\partial \phi^*}{\partial p^*} \kappa^{-1} \right),
$$

$$
A^*_s = -\frac{1}{2} \kappa^{-1} \left( \frac{\partial \phi^*}{\partial p^*} \frac{\partial \phi^*}{\partial z^*} \right),
$$

and

$$
A^*_n = \frac{1}{2} \left( 1 - \frac{\partial \phi^*}{\partial z^*} \frac{\partial \phi^*}{\partial z^*} \right).
$$

In the same way, we have the decomposition

$$
\frac{\partial \text{Tr} A^*}{\partial q^*} = \frac{\partial \text{Tr} A^*}{\partial p^*} \kappa^{-1} \Pi^* + \frac{\partial \text{Tr} A^*}{\partial z^*} N^*.
$$

with $\text{Tr} A^* = \text{Tr}(A^*_t) + A^*_n$.

Now to write the equilibrium equations onto $T_{p^*}\omega^* \oplus \mathbb{R}N^*$, let us decompose as well the applied forces as follows:

$$
f^* = f^*_t + f^*_n N^* \quad \text{and} \quad g^* = g^*_t + g^*_n N^*,
$$

where $f^*_t$, $g^*_t$ and $f^*_n$, $g^*_n$ denote, respectively the tangential and the normal forces acting in the deformed configuration.

Finally, decomposing the divergence of the endomorphism $A^*$ onto $T_{p^*}\omega^* \oplus \mathbb{R}N^*$ (see [18]), the equilibrium equations (4) can be written as

$$
2\mu \left[ \text{div}^*(\kappa^{-1} A^*_t) - \text{div}^*(\kappa^{-1}) A^*_t - \hat{A}^*_s \kappa^{-1} C^* \\
- \text{Tr}(\kappa^{-1} C^*) \hat{A}^*_s + \frac{\partial \hat{A}^*_s}{\partial z^*} \right] \\
+ \lambda \left( \frac{\partial \text{Tr}(A^*_t)}{\partial p^*} \kappa^{-1} + \frac{\partial \hat{A}^*_n}{\partial p^*} \kappa^{-1} = -f^*_t, \right. \tag{7}
$$

$$
2\mu [\text{div}^*(\kappa^{-1} A^*_t) - \text{div}^*(\kappa^{-1}) A^*_t] \\
+ \text{Tr}(\kappa^{-1} C^* A^*_n) - A^*_n \text{Tr}(\kappa^{-1} C^*)] \\
+ \lambda \left( \frac{\partial \text{Tr}(A^*_t)}{\partial z^*} + (\lambda + 2\mu) \frac{\partial A^*_n}{\partial z^*} = -f^*_n, \right. \tag{8}
$$

where $\text{div}^*$ denotes the two-dimensional divergence on the middle surface $\omega^*$.

3.2. Decomposition of the boundary conditions

A difficulty inherent to this Eulerian approach is to take into account the boundary conditions on the upper and lower faces. Indeed as the thickness of the shell in its deformed configuration is not constant, the normal $n^*_\pm$ to the upper and lower faces $\Gamma^*_\pm$ does not coincide with the normal $N^*$ to the middle surface $\omega^*$. As the stress tensor $\sigma^*$ is decomposed onto $T_{p^*}\omega^* \oplus \mathbb{R}N^*$, we have to decompose in the same way the normals $n^*_\pm$ to write the boundary conditions on $\Gamma^*_\pm$.

To do this, let us consider $q^*_\pm = p^* + h^*_\pm N^*$ a current point of the upper face $\Gamma^*_+$. The linear tangent mapping
to \( p^* \mapsto q^* \) which transforms the tangent plane \( T_{p^*} \omega^* \) into the tangent plane \( T_{q^*} \Gamma^*_+ \) is defined as follows:

\[
B^* = \frac{\partial q^*_+}{\partial p^*} = (I_{T_{q^*}} - h^*_+ C^*) + N^* \frac{\partial h^*_+}{\partial p^*},
\]

where \( \frac{\partial h^*_+}{\partial p^*} \) is a linear form and

\[
N^*(\frac{\partial h^*_+}{\partial p^*}) = N^* \otimes \text{grad}(h^*_+).
\]

Using an equivalent matricial notation in \( T_{p^*} \omega^* \oplus \mathbb{R} N^* \), we get

\[
B^* = \begin{pmatrix} \kappa^*_+ & \frac{\partial h^*_+}{\partial p^*} \end{pmatrix}
\]

with \( \kappa^*_+ = I_{T_{q^*}} - h^*_+ C^* \). Now let us search the normal direction \( \tilde{n}^*_+ \) to the upper face \( \Gamma^*_+ \) in the form

\[
\tilde{n}^*_+ = N^* + w^*
\]

with \( w^* \in T_{p^*} \omega^* \). The orthogonality condition \( \tilde{n}^*_+ B^* = 0 \) leads to

\[
\tilde{n}^*_+ B^* + \tilde{w}^* B^* = 0.
\]

According to the expression of \( B^* \), we get

\[
\tilde{n}^*_+ B^* = \frac{\partial h^*_+}{\partial p^*}.
\]

Thus, we obtain

\[
w^* = -\kappa^*_+^{-1} \frac{\partial h^*_+}{\partial p^*}.
\]

Finally norming the vector \( \tilde{n}^*_+ \), we get the following expression of the unit normal \( n^*_+ \) to the upper face \( \Gamma^*_+ \):

\[
n^*_+ = \frac{1}{\sqrt{1 + \frac{\partial h^*_+}{\partial p^*} \kappa^*_+^{-2} \frac{\partial h^*_+}{\partial p^*}}} \left[ N^* - \kappa^*_+^{-1} \frac{\partial h^*_+}{\partial p^*} \right]
\]

with

\[
\kappa^*_+ = I_{T_{q^*}} - h^*_+ C^*.
\]

A similar computation on the lower face with \( q^*_- = p^* - h^*_- N^* \) as a current point leads to

\[
n^*_- = \frac{1}{\sqrt{1 + \frac{\partial h^*_-}{\partial p^*} \kappa^*_-^{-2} \frac{\partial h^*_-}{\partial p^*}}} \left[ -N^* - \kappa^*_-^{-1} \frac{\partial h^*_-}{\partial p^*} \right]
\]

with

\[
\kappa^*_- = I_{T_{q^*}} + h^*_- C^*.
\]

Accordingly, the boundary conditions on the upper and lower faces can be written

\[
\begin{align*}
\hat{\lambda} \text{Tr}(A^*_+ I_{T_{q^*}} - \omega^*) + \hat{\gamma} A^*_+ I_{T_{q^*}} - \omega^*) & n^*_+ \\
+ [\hat{\gamma} A^*_- + \hat{\lambda} \text{Tr}(A^*_-)] n^*_- &= g^*_+ \quad \text{on} \quad \Gamma^*_+, \\
\\hat{\gamma} A^*_- n^*_+ + \hat{\lambda} \text{Tr}(A^*_-) n^*_- &= g^*_- \quad \text{on} \quad \Gamma^*_-, 
\end{align*}
\]

with

\[
n^*_+ = \frac{-\kappa^*_+^{-1} \frac{\partial h^*_+}{\partial p^*}}{\sqrt{1 + \frac{\partial h^*_+}{\partial p^*} \kappa^*_+^{-2} \frac{\partial h^*_+}{\partial p^*}}}
\]

and

\[
n^*_- = \frac{\pm 1}{\sqrt{1 + \frac{\partial h^*_-}{\partial p^*} \kappa^*_-^{-2} \frac{\partial h^*_-}{\partial p^*}}}
\]

We recall that the boundary condition on the free lateral surface is given by \( \sigma^* n^* = 0 \) on \( \Gamma^*_2 \), where \( n^* \) denotes the unit normal to \( \Gamma^*_+ \). However, the asymptotic expansion of this boundary condition requires to express the normal \( n^* \) with respect to the unit external normal \( v^* \) to \( \gamma^* = \hat{\gamma} \omega^* \). To do this we have the following proposition:

**Proposition 2.** Let \( \omega^* \) be a connected surface imbedded in \( \mathbb{R}^3 \). Let us consider the shell which occupies the domain

\[
\Omega^* = \{ q^* = p^* + z^* N \quad \text{where} \quad p^* \in \omega^* \quad \text{and} \quad -h^*_-(p^*) \leq z^* \leq h^*_+(p^*) \}
\]

Then the unit external normal \( n^* \) to the lateral surface \( \Gamma^*_+ \) is given by

\[
n^* = \frac{1}{\| \kappa^*^{-1} v^* \|} \kappa^*^{-1} v^*
\]

with \( \kappa^* = I_{T_{q^*}} - z^* C^* \) and where \( I_{T_{q^*}} \) denotes the identity of \( T_{q^*} \).

The proof of this proposition can be found in [24] in the particular case of a shell with constant thickness. Its generalization to a variable thickness does not constitute any difficulty. Thus, the boundary condition on
the free lateral surface \( I^* \) can be written as
\[
\sigma^* \kappa^{-1} v^* = 0 \text{ and decomposed into }
\]
\[
[\lambda \text{ Tr}(A^*_{rr})I_{T^{rr}_{r,0}} + 2\mu A^*_{rr} + \lambda A^*_{n}I_{T^{nn}_{r,0}}] \kappa^{-1} v^* = 0 \text{ on } I^* \text{,}
\]
\[
2\mu A^*_{n} \kappa^{-1} v^* = 0 \text{ on } I^* \text{.} \tag{15}
\]

4. Dimensional analysis and one-scale problem

Let us define the following dimensionless physical data and unknowns of the problem:
\[
p = \frac{p^*}{L_0}, \quad \phi = \frac{\phi^*}{C^*}, \quad C = \frac{C^*}{C}, \quad f_n = \frac{f_{n}^*}{f_{nr}}, \quad g_n = \frac{g_{n}^*}{g_{nr}}, \quad f_t = \frac{f_t^*}{f_{tr}}, \quad g_t = \frac{g_t^*}{g_{tr}}, \quad h^- = \frac{h_{n}^*}{h_0}, \quad h_+ = \frac{h_{n}^*}{h_0},
\]
where the variables indexed by \( r \) are the reference ones. The new variables which appear without a star are dimensionless. In particular, \( C_r = \| C^* \|_{\infty} \) denotes the maximum curvature of the shell in the deformed configuration.

Let us notice that, as \( \phi^* (q^*) \) denotes the position of the point \( q_0^* \) of the initial configuration (an origin has been chosen once and for all), we have naturally \( \phi_r = L_0 \). This choice of \( \phi_r = L_0 \) does not constitute any assumption on the order of magnitude of the displacements.

The dimensional analysis of the equilibrium equations (7) and (8) leads to the dimensionless equations in \( \Omega = \{ p + z \epsilon; \ p \in \omega; z \in ] - h_-, h_+ [ \} : \)
\[
\epsilon^2 \left\{ 2 \text{ div} (\kappa^{-1} A) - 2 \text{ div} (\kappa^{-1}) A + \beta \frac{\partial \text{ Tr}(A t)}{\partial p} \kappa^{-1} \right\}
\]
\[
+ 2 \frac{\partial A_t}{\partial z} + \beta \frac{\partial A_n}{\partial p} \kappa^{-1} = - \epsilon \frac{\partial f_t}{\partial t}, \tag{16}
\]
\[
2 \epsilon^2 (h_0 C_r) \text{ Tr}(\kappa^{-1} C A_t) + \epsilon^2 \left\{ 2 \text{ div} (\kappa^{-1} A_t) - 2 \text{ div} (\kappa^{-1}) A + \beta \frac{\partial \text{ Tr}(A t)}{\partial p} \kappa^{-1} \right\}
\]
\[
- 2 \text{ div} (\kappa^{-1}) A + \beta \frac{\partial \text{ Tr}(A t)}{\partial z} \right\}
\]
\[
- 2 h_0 C_r A_n \text{ Tr}(\kappa^{-1} C)
\]
\[
+ (2 + \beta) \frac{\partial A_n}{\partial z} = - \epsilon \frac{\partial f_n}{\partial n} \tag{17}
\]
with \( \beta = \lambda / \mu, \ \kappa = I_{T_{0}} - (h_0 C_r) \epsilon C \) and
\[
A_t = \frac{1}{2} \left( I_{T_{0}} - \kappa^{-1} \frac{\partial ^2 \phi}{\partial p} \frac{\partial \phi}{\partial p} \kappa^{-1} \right),
\]
\[
A_n = - \frac{1}{2} \kappa^{-1} \frac{\partial ^2 \phi}{\partial p} \frac{\partial \phi}{\partial p},
\]
\[
A_n = \frac{1}{2} \left( \epsilon^2 - \frac{\partial ^2 \phi}{\partial z} \frac{\partial \phi}{\partial z} \right). \tag{18}
\]

The boundary conditions (11), (12), (15) and the clamping condition become
\[
\beta A_n n_\pm + 2 A_n n_{3\pm} + \epsilon^2 [\beta \text{ Tr}(A_t) I_{T_{0} c} + 2 A_t] n_\pm
\]
\[
= \epsilon^2 g_t g_t^\pm \text{ on } \Gamma_\pm, \tag{19}
\]
\[
(2 + \beta) A_n n_{3\pm} + 2 \epsilon^2 A_n n_\pm + \epsilon^2 \beta \text{ Tr}(A_t) n_{3\pm}
\]
\[
= \epsilon^2 g_n g_n^\pm \text{ on } \Gamma_\pm, \tag{20}
\]
\[
\phi = I \text{ on } \Gamma_1,
\]
\[
[\epsilon^2 (\beta \text{ Tr}(A_t) I_{T_{0} c} + 2 A_t)
\]
\[
\quad + \beta A_n I_{T_{0} c}] \kappa^{-1} v = 0 \text{ on } \Gamma_2, \tag{21}
\]
where
\[
n_{3\pm} = \frac{-1}{\sqrt{1 + \epsilon^2 \frac{\partial h_{\pm}}{\partial p} \kappa^{-2} \frac{\partial h_{\pm}}{\partial p}}},
\]
\[
n_{3\pm} = \frac{\pm 1}{\sqrt{1 + \epsilon^2 \frac{\partial h_{\pm}}{\partial p} \kappa^{-2} \frac{\partial h_{\pm}}{\partial p}}},
\]
\[
\kappa_+ = I_{T_{0}} - (h_0 C_r) h_+ C \text{ and } \kappa_- = I_{T_{0}} - (h_0 C_r) h_- C. \tag{22}
\]
Thus, this dimensional analysis makes naturally appear the following dimensionless numbers which characterize the problem:

(i) The shape ratio \( \epsilon = h_0 / L_0 \) characterizes the relative thickness of the shell in its initial configuration. For thin shells, we have \( \epsilon \ll 1 \).
(ii) The shape factor \( C = h_0 C_r \) characterizes the curvature of the middle surface \( o^* \) of the shell in its deformed configuration.

(iii) The force ratios \( F_t = h_0 f_t/\mu \), \( F_n = h_0 f_n/\mu \) and \( G_t = g_t/\mu \), \( G_n = g_n/\mu \) are similar to those introduced in \([14–16,18]\). They represent the ratio of the resultant on the thickness of the body forces (respectively the ratio of the surface forces) to \( \mu \) considered as a reference stress. They characterize the order of magnitude of the \( \text{Eulerian forces} \) acting in the deformed configuration and depend only on known measurable quantities of the problem.

To apply the standard method of asymptotic expansion, the problem must be reduced to a one-scale problem where \( \varepsilon = h_0/L_0 \) is chosen as the reference perturbation parameter. The other dimensionless numbers must be linked to \( \varepsilon \).

First, within the framework of large displacements which are the subject of this paper, we allow a priori the shell to be strongly bent in its deformed configuration. Thus we fix a priori \( L_0 C_r = O(1) \) or equivalently \( h_0 C_r = O(\varepsilon) \). Thus, in what follows we will consider that \( C = h_0 C_r = \varepsilon \). It will be always possible to consider another reference scale for the deformed curvature if we need.

On the other hand, like in usual elasticity problems, we consider here only the gravity as body force acting on the shell.\( ^3 \) To determine the order of magnitude of the force ratios \( F_t \) and \( F_n \) associated to the gravity, let us consider the following example: a thin steel shell of \( L_0 = 1 \) m diameter and \( h_0 = 10^{-2} \) m thickness, whose Young’s modulus, Poisson’s ratio and voluminal mass are, respectively, \( E = 2.1 \times 10^{11} \) Pa, \( v = 0.285 \) and \( \rho = 7800 \) kg/m\(^3\). As the shell is a priori strongly bent in its deformed configuration, the normal and tangential components of the weight play a symmetric role and we have \( F_t = F_n = \rho gh_0/\mu = 10^{-8} \). As \( \varepsilon = h_0/L_0 = 10^{-2} \), we have \( F_t = F_n = \varepsilon^4 \). Therefore, we will consider here a level of body forces such as \( F_t = F_n = \varepsilon^4 \).

Finally to obtain large displacements, we consider a severe level of surface forces \( G_t = G_n = \varepsilon \).

5. Asymptotic expansion of equations

Once reduced to a one-scale problem, following the method of asymptotic expansion, we postulate that the mapping \( \phi \) admits a formal series expansion
\[
\phi = \phi(\varepsilon) = \phi^0 + \varepsilon \phi^1 + \varepsilon^2 \phi^2 + \cdots.
\]

The expansion of \( \phi \) then implies an expansion of \( A_t, A_s \) and \( A_n \) with respect to \( \varepsilon \):
\[
\begin{align*}
A_t &= A_t^0 + \varepsilon A_t^1 + \varepsilon^2 A_t^2 + \cdots, \\
A_s &= A_s^0 + \varepsilon A_s^1 + \varepsilon^2 A_s^2 + \cdots, \\
A_n &= A_n^0 + \varepsilon A_n^1 + \varepsilon^2 A_n^2 + \cdots.
\end{align*}
\]

In particular, we get
\[
\begin{align*}
A_t^0 &= \frac{1}{2} \left( I_{T,\psi} - \frac{\partial \phi^0}{\partial \theta} \frac{\partial \phi^0}{\partial \theta} \right), \\
A_s^0 &= -\frac{1}{2} \frac{\partial \phi^0}{\partial \theta} \frac{\partial \phi^0}{\partial \theta} \\
A_n^0 &= -\frac{1}{2} \frac{\partial \phi^0}{\partial \theta} \frac{\partial \phi^0}{\partial \theta}, \\
A_t^1 &= -\frac{1}{2} \left( \frac{\partial \phi^0}{\partial \theta} \frac{\partial \phi^1}{\partial \theta} + \frac{\partial \phi^1}{\partial \theta} \frac{\partial \phi^0}{\partial \theta} + \varepsilon C \frac{\partial \phi^0}{\partial \theta} \frac{\partial \phi^0}{\partial \theta} \right), \\
A_s^1 &= -\frac{1}{2} \frac{\partial \phi^0}{\partial \theta} \frac{\partial \phi^1}{\partial \theta}, \\
A_n^1 &= -\frac{1}{2} \frac{\partial \phi^0}{\partial \theta} \frac{\partial \phi^1}{\partial \theta} \\
A_t^2 &= \frac{1}{2} \left( 1 - 2 \frac{\partial \phi^0}{\partial \theta} \frac{\partial \phi^2}{\partial \theta} - \frac{\partial \phi^1}{\partial \theta} \frac{\partial \phi^1}{\partial \theta} \right). \\
\end{align*}
\]

To make the asymptotic expansion of Eqs. (16) and (17) and boundary conditions (19)–(21), all the terms must be developed with respect to \( \varepsilon \). On one hand, as \( \kappa = \kappa(\varepsilon) = I_{T,\psi} + \varepsilon z \), we have
\[
\kappa^{-1}(\varepsilon) = I_{T,\psi} + \varepsilon z C + \varepsilon^2 z^2 C^2 + \varepsilon^3 z^3 C^3 + \cdots.
\]

On the other hand, according to expression (22), the normals \( n_{t\pm} \) and \( n_{s\pm} \) admit at the first order the following expansion:
\[
n_{t\pm} = -\frac{\partial h}{\partial \theta} + O(\varepsilon) \quad \text{and} \quad n_{s\pm} = \pm 1 + O(\varepsilon^2).
\]
We then have the following proposition:

**Proposition 3.** For a severe force level \( \mathcal{G}_i = \mathcal{G}_n = \varepsilon \) and \( \mathcal{F}_i = \mathcal{F}_n = \varepsilon^4 \), the leading term \( \phi^0 = \phi^0(p) \) of the expansion of \( \phi \) depends only on \( p \) and is solution of the following non-linear membrane problem:

\[
\text{div}(n^0_i) = - \tilde{p}_i \quad \text{in } \omega, \\
\text{Tr}(n^0_i C) = - p_n \quad \text{in } \omega, \\
\phi^0 = i_{\omega} \quad \text{on } \gamma_1, \\
n^0_i v = 0 \quad \text{on } \gamma_2
\]  
(26)

with

\[
n^0_i = \frac{4\beta}{2 + \beta} \text{Tr}A^0_i I_{T_i \omega} + 4A^0_i, \\
A^0_i = \frac{1}{2} \left( I_{T_i \omega} - \frac{\partial \phi^0}{\partial p} \frac{\partial \phi^0}{\partial p} \right), \\
p_i = g^+_i + g^-_i \quad \text{and} \quad p_n = g^+_n + g^-_n,
\]

where \( g^+_i, g^+_n \) denote, respectively, the values of \( g_i, g_n \) on the upper and lower deformed faces \( T_\pm \), and \( \gamma_1 \cup \gamma_2 = \gamma \) the boundary of \( \omega \).

**Proof.** The proof of this proposition is broken into six steps from (i) to (vi).

(i) \( \phi^0 \) depends on \( p \) only

Let us replace \( \phi \) by its expansion (23) in the equilibrium equations (16) and (17) and the boundary conditions (19)–(21), where the dimensionless numbers have been replaced by \( \bar{\varepsilon} = \varepsilon \), \( \mathcal{F}_i = \mathcal{F}_n = \varepsilon^4 \) and \( \mathcal{G}_i = \mathcal{G}_n = \varepsilon \). We then obtain a sequence of coupled problems \( \mathcal{P}^0, \mathcal{P}^1, \mathcal{P}^2 \ldots \) corresponding, respectively, to the cancellation of the factor of \( \varepsilon^2, \varepsilon^1, \varepsilon^0, \ldots \).

Problem \( \mathcal{P}_0 \) reduces to

\[
\beta \frac{\partial A^0_n}{\partial p} + 2 \frac{\partial A^0_n}{\partial z} = 0 \quad \text{in } \Omega, \\
(2 + \beta) \frac{\partial A^0_p}{\partial z} = 0 \quad \text{in } \Omega, \\
A^0_s = 0 \quad \text{for } z = \pm h_\pm, \\
A^0_n = 0 \quad \text{for } z = \pm h_\pm.
\]  
(27)  
(28)  
(29)  
(30)

Eqs. (28) and (30) imply that \( A^0_n = 0 \) in \( \Omega \) equivalently

\[
\frac{\partial \phi^0}{\partial z} = 0 \quad \text{in } \Omega.
\]

Thus, we obtain

\[
\phi^0 = \phi^0(p). \quad \text{(31)}
\]

Eqs. (27) and (29) are then trivially satisfied according to expression (24) of \( A^0_i \).

(ii) Expression of \( \phi^1 \)

Taking into account (31), problem \( \mathcal{P}_1 \) reduces to

\[
\frac{\partial A^1_i}{\partial z} = 0 \quad \text{in } \Omega, \\
A^1_i = 0 \quad \text{for } z = \pm h_\pm \quad \text{(32)}
\]

and implies that \( A^1_s = 0 \) or equivalently

\[
\frac{\partial \phi^1}{\partial p} = 0 \quad \text{in } \Omega. \quad \text{(34)}
\]

So we get

\[
\frac{\partial \phi^1}{\partial z} = \zeta(z, p) \tilde{N}_0, \quad \text{(35)}
\]

where \( \tilde{N}_0 \) denotes the unit normal to the surface \( \phi^0(\omega) \). For the moment nothing assures that \( \phi^0(\omega) \) corresponds to the middle surface \( \omega_0 \) of the initial configuration. We will prove at point (iv) that we have effectively \( \phi^0(\omega) = \omega_0 \) and \( \tilde{N}_0 = N_0 \), where \( N_0 \) denotes the unit normal to \( \omega_0 \).

(iii) Membrane equations

The cancellation of the factor of \( \varepsilon^3 \) leads to problem \( \mathcal{P}_2 \) which reduces to

\[
2 \text{div}A^0_i + \beta \frac{\partial \text{Tr}A^0_i}{\partial p} + \beta \frac{\partial A^2_n}{\partial p} + 2 \frac{\partial A^2_s}{\partial z} = 0 \quad \text{in } \Omega, \quad \text{(36)}
\]

\[
(2 + \beta) \frac{\partial A^2_s}{\partial z} + \beta \frac{\partial \text{Tr}A^0_i}{\partial z} = 0 \quad \text{in } \Omega, \quad \text{(37)}
\]

\[
\pm 2A^2_s - (\beta A^2_s I_{T_i \omega} + \beta \text{Tr}(A^0_i)I_{T_i \omega} + 2A^0_i) \frac{\partial h_s}{\partial p} = g^+_i \quad \text{for } z = \pm h_\pm, \quad \text{(38)}
\]

\[
(2 + \beta) A^2_n + \beta \text{Tr}A^0_i = 0 \quad \text{for } z = \pm h_\pm. \quad \text{(39)}
\]
Eqs. (37) and (39) lead to
\[ A_n^2 = \frac{\beta}{2 + \beta} \text{Tr} A_t^0 \quad \text{in } \Omega, \tag{40} \]

where
\[ A_t^0 = \frac{1}{2} \left( I_{T_{\nu \phi}} - \frac{\partial \phi^0}{\partial \nu} \frac{\partial \phi^0}{\partial p} + \frac{\partial \phi^0}{\partial \nu} \right) \tag{41} \]
denotes the metric variation due to \( \phi^0 \) and calculated in the deformed configuration.

It is then possible to compute \( \zeta(p, z) \). Indeed as
\[ A_n^2 = \frac{1}{2} \left( 1 - \left\| \frac{\partial \varphi^1}{\partial z} \right\|^2 \right) \]
we get according to (35)
\[ \left\| \frac{\partial \varphi^1}{\partial z} \right\|^2 = \zeta^2(p) = 1 + \frac{2\lambda}{\lambda + 2\mu} \text{Tr} A_t^0, \]

Thus \( \varphi^1 \) is given by
\[ \varphi^1(p, z) = u^1(p) + z\zeta(p)N_0, \]
where \( \zeta \) is a function which depends only on \( p \in \Omega \) and characterizes the compression upon the thickness. \(^4\)

Now replacing \( A_n^2 \) by its expression (40) (respectively (39)) in Eqs. (36) and (38), we get
\[ 2 \text{div} A_t^0 + \frac{2\beta}{2 + \beta} \frac{\partial \text{Tr} A_t^0}{\partial p} + 2 \frac{\partial A_t^0}{\partial z} = 0 \quad \text{in } \Omega, \tag{43} \]

\[ \pm 2A^e_z - \left( \frac{2\beta}{2 + \beta} \text{Tr} A_t^0 I_{T_{\nu \phi}} + 2A_t^0 \right) \frac{\partial h^\pm}{\partial p} = g^\pm \quad \text{for } z = \pm h^\pm. \tag{44} \]

Taking into account the boundary conditions (44), and integrating Eq. (43) upon the thickness (between \( -h^- \) and \( h^+ \)), we obtain
\[ h \text{div}(n_t^0) + \frac{\partial h}{\partial p} n_t^0 = \text{div}(h h_t^0) = -\tilde{p}_t \quad \text{in } \Omega, \tag{45} \]

where
\[ n_t^0 = \frac{4\beta}{2 + \beta} \text{Tr} A_t^0 I_{T_{\nu \phi}} + 4A_t^0, \]

\[ h = h^+ + h^- \quad \text{and} \quad p_t = g^+_n + g^-_n. \]

To obtain the second membrane equation, let us explicit problem \( \varphi_3 \) whose second equation and boundary condition reduce to
\[ 2 \text{Tr}(C\varphi_3^1) - 2A^2_n \text{Tr} C + \beta \frac{\partial \text{Tr} A_t^0}{\partial z} \]
\[ + (2 + \beta) \frac{\partial A^3_n}{\partial z} = 0 \quad \text{in } \Omega, \tag{46} \]

\[ (2 + \beta)A^3_n + \beta \text{Tr} A_t^0 = \pm g^\pm_n \quad \text{for } z = \pm h^\pm. \tag{47} \]

Replacing \( A_n^2 \) by its expression (40) and using (47), an integration upon the thickness of Eq. (46) leads to
\[ \frac{h^+ + h^-}{2} \left[ \frac{4\beta}{2 + \beta} \text{Tr} A_t^0 \text{Tr} C + 4 \text{Tr}(A_t^0 C) \right] = -p_n \]

or equivalently to
\[ \text{Tr}(h h_t^0 C) = -p_n \tag{48} \]

with \( p_n = g^+_n + g^-_n \).

(iv) Interpretation of \( \zeta \)

Up to now, with this Eulerian approach the calculations have been made in the deformed configuration. Now we must take into account the fact that the shell has been deformed in a continuous way from a initial configuration which is a data of the problem. To do this we impose that the upper and lower faces of the initial configuration deform onto the upper and lower faces of the shell. \(^6\)

We recall that \( \Omega_0^\pm \) is the initial configuration of the shell, whose middle surface is \( \omega_0^\pm \) and thickness \( 2h_0 \).

\(^4\) This relation has a meaning if \( 1 + 2\lambda/(\lambda + 2\mu) \text{Tr} A_t^0 \) stays positive during the deformation, i.e. if the membrane strains are not too much important. Such a limitation has already been noticed in [25] within the framework of a Lagrangian approach. This limitation seems to come from the linearized constitutive law which is used and which limits a priori our study to small strains. However, an explicit calculus for different materials proves that this quantity is positive as long as relative elongations are less than 0.5. After we go out of the domain of elasticity except for particular materials such as rubber-like materials.

\(^5\) The sign of \( \zeta(p) \) is chosen so as to conserve the orientation of the initial middle surface \( \omega_0^\pm \).

\(^6\) This condition is natural. It comes from the non-interpenetration condition of the matter which can be written \( \text{det}(\partial \varphi^*/\partial q^*) > 0 \).
Thus, a point \( q_{0+}^* \) or \( q_{0-}^* \), respectively, of the upper or lower face, is given by
\[
q_{0+}^* = p_0^* + h_0N_0 \quad \text{and} \quad q_{0-}^* = p_0^* - h_0N_0,
\]
where \( p_0^* \in \omega_0^* \) and \( N_0 \) denotes the unit normal to \( \omega_0^* \).

The upper and lower faces of the shell in its initial configuration become the upper and lower faces of the shell after deformation and only if
\[
\phi^*(p^*, h_0^*) = q_{0+}^* = p_0^* + h_0N_0,
\]
\[
\phi^*(p^*, -h_0^*) = q_{0-}^* = p_0^* - h_0N_0.
\]
Writing this condition in a dimensionless form, we get
\[
\phi(p, h_+) = \phi^0(p, h_+) + \varepsilon \phi^1(p, h_+) + \cdots = p_0 + \varepsilon N_0,
\]
\[
\phi(p, -h_) = \phi^0(p, -h_) + \varepsilon \phi^1(p, -h_) + \cdots = p_0 - \varepsilon N_0.
\]
Identifying the terms of the same power with respect to \( \varepsilon \), we obtain
\[
\phi^0(p, h_+) = \phi^0(p, -h_) = p_0, \quad (49)
\]
\[
\phi^1(p, h_+) = N_0, \quad (50)
\]
\[
\phi^1(p, -h_) = -N_0. \quad (51)
\]
Conditions (49) assures that \( \phi^0(\omega) = \omega_0 \), where \( \omega_0 \) denotes the dimensionless middle surface of the shell in its reference configuration. Thus, we have \( N_0 = N_0 \) and expression (42) of \( \phi^1 \) becomes
\[
\phi^1(p, z) = u^1(p) + z\zeta(p)N_0,
\]
\[
\zeta(p) = \sqrt{1 + \frac{2\lambda}{\lambda + 2\mu} \text{Tr} A_t^0}. \quad (52)
\]

Now combining expressions (50) and (51), according to the previous expression of \( \phi^1 \) we obtain
\[
\frac{h_+ + h_-}{2}\zeta(p) = h\zeta(p) = 1 \quad (53)
\]
or equivalently \( (h^+/h_0)\zeta(p) = 1 \). Therefore, \( \zeta(p) \) is directly linked to the thickness variation during the deformation.\(^7\)

\(^7\) For a severe level of surface forces \( \gamma_n = \gamma_i = \varepsilon \), the variation of thickness \( (h^+ - h_0)/h_0 = (1/\zeta(p)) - 1 \) directly depends on the membrane variation \( A_t^0 \).

(v) **Linearization for small strains**

The linearization of the expression of \( \zeta(p) \) for small membrane strains \( A_t^0 \) leads to
\[
h = \frac{h^*}{h_0} = 1 - \frac{\lambda}{\lambda + 2\mu} \text{Tr} A_t^0 + O((A_t^0)^2)
\]
and
\[
h n_t^0 = n_t^0 + O((A_t^0)^2).
\]

In the framework of small strains (but large displacements), we have to neglect the terms of the second order with respect to the membrane strain in Eqs. (45) and (48). We then obtain
\[
\text{div}(n_t^0) = -\tilde{p}_t,
\]
\[
\text{Tr}(n_t^0 \lambda) = -p_n
\]
which constitutes the Eulerian equations of Proposition 3.

(vi) **Boundary conditions**

The boundary conditions associated to this Eulerian membrane model are obtained naturally by expansion of the three-dimensional boundary conditions (21).

First, the expansion of the clamped condition on \( I_1 \) leads immediately to \( \phi^0 = \iota_\omega \) on \( \gamma_1 \).

On the other hand, the expansion of the boundary condition on \( I_2 \) gives us at the leading order
\[
\varepsilon^2(2\lambda \text{Tr}(A_t^0)I_{\varepsilon^2} + 2A_t^0 + \beta A^2_n I_{\varepsilon^2})v = O(\varepsilon^3)
\]
\[
= 0 \quad \text{on} \quad \gamma_2.
\]

Taking into account (40), we obtain \( n_t^0 v = 0 \) on \( \gamma_2 \) according to the definition of \( n_t^0 \). This concludes the proof of Proposition 3.

6. **Back to the Lagrangian variables**

To compare the Eulerian equations of membrane obtained at Proposition 3 with other Lagrangian membrane equations which exist in the literature, we have to go back to the Lagrangian variables. This pull-back of Eulerian equilibrium equations on \( \omega_0 \) can be made using the mapping
\[
\phi^0 : \omega \to \omega_0,
\]
\[
p \mapsto p_0 = \phi^0(p)
\]
or the inverse mapping \( \psi^0 = (\phi^0)^{-1} : \omega_0 \to \omega. \)
6.1. Pull-back of the Eulerian membrane stress tensor

First, let us define the equivalent of the second tensor of Piola-Kirchhoff $N^0_t$ obtained by pulling back $n^0$ as follows:

$$N^0_t = \sqrt{g} \left( \frac{\partial p}{\partial p_0} \right)^{-1} n^0_t \left( \frac{\partial p}{\partial p_0} \right)^{-1},$$

where $\frac{\partial p}{\partial p_0} : T_{p_0} \omega_0 \to T_p \omega$ denotes the linear tangent mapping to the application $p_0 \mapsto p$ and $g = \det((\partial p/\partial p_0)\partial p/\partial p_0)$ the determinant of the metric tensor of the deformed middle surface $\omega$.

We then have the following proposition:

**Proposition 4.** For small strains, $N^0_t$ can be linked to the classical Lagrangian membrane strain tensor

$$E^0_t = \frac{1}{2} \left( \frac{\partial p}{\partial p_0} \frac{\partial p}{\partial p_0} - I_{T_{p_0} \omega_0} \right)$$

characterizing the variation of metric calculated in the initial configuration. We have

$$N^0_t = \frac{4\beta}{2 + \beta} (\text{Tr} E^0_t) I_{T_{p_0} \omega_0} + 4E^0_t + O((E^0_t)^2).$$

**Proof.** According to the definition of

$$A^0_t = \frac{1}{2}(I_{T_{p_0} \omega} - (\partial \varphi/\partial p)\partial \varphi/\partial p)$$

where $\varphi^0(p) = p_0$, we have

$$\frac{\partial p}{\partial p_0} A^0_t \frac{\partial p}{\partial p_0} = E^0_t.$$

As the linear tangent mapping to $p_0 \mapsto p$ can be inverted, we get

$$A^0_t = \left( \frac{\partial p}{\partial p_0} \right)^{-1} E^0_t \left( \frac{\partial p}{\partial p_0} \right)^{-1}.$$

On the other hand, as $\partial p/\partial p_0$ is invertible, $(\partial p/\partial p_0)$

$\partial p/\partial p_0$ is invertible as well and we have

$$\left( \frac{\partial p}{\partial p_0} \right)^{-1} \left( \frac{\partial p}{\partial p_0} \right)^{-1} = (I_{T_{p_0} \omega_0} + 2E^0_t)^{-1}$$

according to expression (55) of $E^0_t$. Then expression (54) of $N^0_t$, can be written

$$N^0_t = \sqrt{\det(I_{T_{p_0} \omega_0} + 2E^0_t)} \left[ \frac{4\beta}{2 + \beta} \text{Tr}(E^0_t(I_{T_{p_0} \omega_0} + 2E^0_t)^{-1}) + 4(E^0_t)^2 \right].$$

The linearization of this expression for small membrane strains, where $\|E^0_t\| \ll 1$, leads to

$$N^0_t = \frac{4\beta}{2 + \beta} (\text{Tr} E^0_t) I_{T_{p_0} \omega_0} + 4E^0_t + O((E^0_t)^2)$$

which corresponds at the first order to the classical expression of the non-linear membrane stress tensor obtained from asymptotic expansion of the three-dimensional Lagrangian equilibrium equations [24] (see also [16,25] in the particular case of plates).

6.2. Weak formulation associated

Now the pull-back of Eulerian equations (26) of Proposition 3 must be made with the weak formulation associated. The equations so obtained will be compared to those obtained in [24] with a Lagrangian approach.

If we assume that the mapping $\varphi^0$ is smooth enough, the Eulerian equations of Proposition 3 can be written in an equivalent weak formulation. To do this let us define the following spaces of admissible mappings and displacements:

$$A(\omega) = \{ \varphi : \omega \to \mathbb{R}^3, "smooth", \varphi = i_\omega \text{ on } \gamma_1 \},$$

$$V(\omega) = \{ v = v_t + v_\omega N : \omega \to \mathbb{R}^3, "smooth", v = 0 \text{ on } \gamma_1 \},$$

where $i_\omega$ denotes the identity mapping on $\omega$.

Then the Eulerian equations of membrane (26) can be written in the following equivalent weak formulation:

Find $\varphi^0 \in A(\omega)$ such that $\forall v \in V(\omega)$

$$\int_{\omega} \text{Tr}(n^0 \Pi \frac{\partial v}{\partial p}) \, d\omega = \int_{\omega} \tilde{p}v \, d\omega$$

(58)
with the notations \( p = p_t + p_n N \) and \( \Pi(\partial v / \partial p) = (\partial v / \partial p) - v_n C \), where \( \Pi \) denotes the orthogonal projection onto the middle surface \( \omega \), and \( \partial v / \partial p \) the covariant derivative of the tangent vector field \( v_t \) along the surface \( \omega \).

6.3. Global pull-back of equations

Let us define the space of admissible displacements on \( \omega_0 \)
\[ V(\omega_0) = \{ v^0 : \omega_0 \to \mathbb{R}^3 \text{ "smooth", } v^0 = 0 \text{ on } \gamma_{01} \}. \]
As the shell is clamped on the portion \( \gamma_{01} \) of its lateral surface, \( \gamma_1 \) and \( \gamma_{01} \) coincide.

To pull-back as a whole the Eulerian membrane equations (26) using the weak formulation (58), it is convenient to introduce the divergence \( \text{div}_{\omega_0} \) of a field of operators \( A_{\omega_0} : \omega_0 \to \mathcal{L}(\mathbb{R}^3, T_{\omega_0}) \) as follows (see [23]):
\[ \text{div}_{\omega_0}(A_{\omega_0})W = \text{div}(A_{\omega_0}W) - \text{Tr} \left( A_{\omega_0} \frac{\partial W}{\partial p_0} \right) \]
for all vector field \( W : \omega_0 \to \mathbb{R}^3 \) defined on \( \omega_0 \), where div denotes the classical two-dimensional divergence on \( \omega_0 \). The divergence \( \text{div}_{\omega_0} \) generalizes the classical two-dimensional divergence of a field of endomorphisms of \( T_{\omega_0} \). We then have the following proposition:

**Proposition 5.** The pull-back on \( \omega_0 \) of Eqs. (58) leads to the following Lagrangian equilibrium equations which can be expressed in terms of the inverse mapping \( \psi^0 = (\varphi^0)^{-1} \):
\[ \text{div}_{\omega_0} \left( N^0_t \frac{\partial \psi^0}{\partial p_0} \right) = -p^0 \quad \text{in } \omega_0, \]
\[ \psi^0 = i_{\omega_0} \quad \text{on } \gamma_{01}, \]
\[ N^0_r v_0 = 0 \quad \text{on } \gamma_{02} \quad \text{(59)} \]
with
\[ N^0_t = \frac{4\beta}{2 + \beta} \text{Tr} E^0_t I_{\bar{r}_0 \omega_0} + 4E^0_t, \]
\[ E^0_t = \frac{1}{2} \left( \frac{\partial \psi^0}{\partial p_0} \frac{\partial \psi^0}{\partial p_0} - I_{\bar{r}_0 \omega_0} \right), \]
\[ p^0 = \sqrt{g} p \circ \psi^0 \]
and where \( v_0 \) denotes the unit external normal to \( \gamma_0 \).

**Proof.** The pull back on \( \omega_0 \) of Eq. (58) leads to
\[ \int_{\omega_0} \text{Tr} \left( n^0_t \Pi \frac{\partial v^0}{\partial p_0} \frac{\partial p_0}{\partial p} \right) \sqrt{g} d\omega_0 \]
\[ = \int_{\omega_0} p^0 v^0 d\omega_0 \quad \forall v^0 = v \circ \psi^0 \in V(\omega_0) \]
with \( p^0 = \sqrt{g} p \circ \psi^0 \). According to definition (54) of \( N^0_r \), we get
\[ \int_{\omega_0} \text{Tr} \left( N^0_r \frac{\partial p}{\partial p} \Pi \frac{\partial v^0}{\partial p_0} \right) d\omega_0 = \int_{\omega_0} p^0 v^0 d\omega_0 \]
\[ \forall v^0 = v \circ \psi^0 \in V(\omega_0). \]
Using the divergence \( \text{div}_{\omega_0} \) previously defined, as \( v^0 = 0 \) on \( \gamma_{01} \), we obtain
\[ \int_{\omega_0} -\text{div}_{\omega_0} \left( N^0_r \frac{\partial p}{\partial p} \Pi \frac{\partial v^0}{\partial p_0} \right) v^0 d\omega_0 + \int_{\gamma_{02}} v^0 \Pi \frac{\partial p}{\partial p} N^0_r v_0 d\gamma_0 \]
\[ = \int_{\omega_0} p^0 v^0 d\omega_0 \quad \forall v^0 \in V(\omega_0), \]
where we have used the following equality \( \int_{\omega_0} N^0_r \left( \frac{\partial p}{\partial p} \Pi \frac{\partial v^0}{\partial p_0} \right) v^0 d\omega_0 = \int_{\omega_0} p^0 \Pi \frac{\partial p}{\partial p} N^0_r v_0 d\omega_0 \).
As \( p = \psi^0(p_0) \), this weak formulation leads naturally to the Lagrangian equilibrium equations (59) and to the boundary conditions \( \frac{\partial p}{\partial p} \big|_{\partial \gamma_{02}} = N^0_r v_0 = 0 \) on \( \gamma_{02} \). As \( \frac{\partial p}{\partial p} \big|_{\partial \gamma_{02}} \) is invertible, the boundary condition on \( \gamma_{02} \) reduces to:
\[ N^0_r v_0 = 0 \quad \text{on } \gamma_{02} \quad \text{(60)} \]
Finally, as \( \varphi^0 = i_{\omega_0} \) on \( \gamma_1 \) which coincides with \( \gamma_{01} \), the clamped condition can also be written
\[ \psi^0 = (\varphi^0)^{-1} = i_{\omega_0} \quad \text{on } \gamma_{01}. \square \]

6.4. Comparison with results obtained with a Lagrangian approach

First, let us recall that with the Lagrangian approach developed in [24], the three-dimensional equilibrium equations are decomposed in the initial configuration.
This makes appear explicitly the initial curvature $C_0$ of the shell. Thus it is necessary to distinguish the initially strongly bent shells (corresponding to $C_0, L_0 = 1$ or to $C_0 = h_0 C_{br} = \varepsilon$) from the initially shallow shells (corresponding to $C_0 = \varepsilon^2$), where $C_{br}$ denotes the reference curvature of the shell in its initial configuration.

In contrary, with the Eulerian approach developed here, no distinction concerning the initial curvature of the shell is needed. The non-linear membrane model obtained is general. It is valid in the general case of initially strongly bent shells (where $C_0 = \varepsilon$) which are also called by other authors as “general shells”. The model is of course valid in the particular case of initially shallow shells (where $C_0 = \varepsilon^2$).

Taking into account the previous remark, it is now possible to compare the Lagrangian equations of Proposition 5 with those obtained with a Lagrangian approach in [24].

For small strains, according to Proposition 4, the left member of membrane equations (59) involving $N_0^0$ coincides with those of the non-linear membrane model obtained for strongly bent shells from a Lagrangian approach [24].

The particular case of shallow shells ($C_0 = \varepsilon^2$) can be deduced from Proposition 5 by considering that the initial curvature is very small and tends towards zero. For this, let us introduce the displacement $U_0^0 = V^0 + u^0 N_0$, where $V^0$ and $u^0$ denote, respectively, the tangential and the normal displacement of the middle surface $\omega_0$. Then we have $\psi^0 = i_{\omega_0} + U^0$ and

$$\frac{\partial \psi^0}{\partial p_0} = I_{\omega_0} + \left( \frac{\partial C_0}{\partial p_0} - u^0 C_0 \right) + N_0 \left( \frac{\partial u^0}{\partial p_0} + \tilde{V}^0 C_0 \right).$$

Then if $C_0$ tends towards zero, the expression of the strain $E_0^0 = \frac{1}{2} ((\partial \psi^0 / \partial p_0) \partial \psi^0 / \partial p_0 - I_{\omega_0})$ reduces to

$$E_0^0 = \frac{1}{2} \left( \frac{\partial V^0}{\partial p_0} + \frac{\partial V^0}{\partial p_0} - \frac{\partial V^0}{\partial p_0} + \frac{\partial V^0}{\partial p_0} \frac{\partial u^0}{\partial p_0} + \frac{\partial u^0}{\partial p_0} \frac{\partial u^0}{\partial p_0} \right)$$

whereas the constitutive relation between $N_0^0$ and $E_0^0$ and the equilibrium equations remains the same. Therefore, the first member of the Lagrangian model so obtained corresponds to the non-linear membrane model obtained for shallow shells ($C_0 = \varepsilon^2$) with a Lagrangian approach [24].

However, there is a profound difference between the Lagrangian and the Eulerian approach. Indeed with the Lagrangian approach, the forces involved are dead and all informations concerning the direction of the forces are lost in the final asymptotic model. In contrary, with the Eulerian approach developed here, it is the components of the pulled-back Eulerian forces which are taken into account. In particular, for large displacements, following forces whose direction varies during the deformation can be taken into account. This is particularly interesting for fluid-structure coupling problems.

7. Particular case of plates

Up to now, we have used the more general notations which are valid if the initial configuration is a shell. However, for a plate in large displacements, we can identify $(\omega_0, \psi^0)$ with the system of coordinates of $\omega$. Then the non-classical divergence $\text{div}_0$ and the equilibrium equations (59) can be decomposed easily onto the tangent plane and the normal to the middle surface as follows:

**Proposition 6.** In local coordinates, the Lagrangian equilibrium equations (59) can be written

$$- [(N^0_1)_{\alpha\beta} g_{\beta\alpha}]_x + (N^0_1)_{\alpha\beta} g_{\beta\alpha} \Gamma^2_{\alpha\beta} = p_0^0, \quad (61)$$

$$(N^0_1)_{\alpha\beta} C_{\alpha\beta} = - p_n^0 \quad (62)$$

where $g_{\alpha\beta}$ denotes the components of the metric tensor of the surface $\omega$, $p^0_\mu$ the covariant components of the tangential forces $p^0_\mu = \sqrt{g} p_\mu \circ \psi^0$ in the local basis of $T_{\omega^0}$ and $p_n^0 = \sqrt{g} p_n \circ \psi^0$ the normal component.

**Proof.** In local coordinates, the Lagrangian equations (59) of Proposition 5 become:

$$\frac{1}{2} (\partial \psi^0 / \partial p_0) \partial \psi^0 / \partial p_0 - I_{\omega_0} = (N^0_1)_{\alpha\beta} a_{\alpha\beta} = - p_n^0, \quad (63)$$

9 This dead loads assumptions (the loads do not depend on the configuration) is necessary to make the asymptotic expansion of Lagrangian three-dimensional equations. 
10 Pressure forces for example.
11 The left member of these equations is similar to those obtained for plates in [16, 25] with a Lagrangian approach. But the expression of the forces is different.
where \( N^a_i \) denotes the contravariant components of the Lagrangian membrane stress tensor \( N_i^0 \) and \( \{a_1, a_2\} \) the covariant basis of the tangent plane \( T_p \omega \).

Separating the tangent from the normal part in Eqs. (63), we obtain
\[
(N^a_i a^a_{j})_x = N^a_i a^a_{j} + N^a_0 a^a_{j, x}
\]
\[
= [N^a_i + N^a_0 \Gamma^a_{a_j}] a^a_{j} + N^a_0 C^a_{a_j} N
\]
\[
= -p^0 a^a_{j} - p^0_n N
\]
which leads to the equilibrium equations
\[
(N^a_i a^a_{j})_x = N^a_0 C^a_{a_j} = -p^0_n N
\]
where \( p^0_n \) denotes the contravariant components of the tangential forces \( p^0 \) in the local basis \( \{a_1, a_2\} \).

Now let us apply the first term in plane equation to a generic tangent vector \( a^a_{\mu} \in T_p \omega \). We obtain
\[
N^a_0 g_{\mu \beta} + N^a_0 \Gamma^a_{a_j} g_{\mu \beta} = -p^0_n g_{\mu \beta} = -p^0_n
\]
or equivalently
\[
N^a_0 g_{\mu \beta} + N^a_0 \Gamma^a_{a_j} g_{\mu \beta} = -p^0_n
\]
(64)
To make appear clearly the divergence operator, let us write the first term of (64) as follows:
\[
N^a_0 g_{\mu \beta} = [N^a_0 \ g_{\mu \beta}, x] - N^a_0 g_{\mu \beta, x}.
\]
Thus, we obtain the equilibrium equations
\[
[N^a_0 \ g_{\mu \beta}, x] + N^a_0 \Gamma^a_{a_j} g_{\mu \beta} = -p^0_n
\]
where
\[
\Gamma^a_{a_j} g_{\mu \beta} = \Gamma^a_{a_j} g_{\mu \beta}
\]
denotes the Christoffel symbols of second type. Then using the second Ricci formula, we get
\[
\Gamma^a_{a_j} g_{\mu \beta} = -\Gamma^a_{a_j \mu} g_{\beta \beta} = -\Gamma^a_{a_j \mu} g_{\beta \beta},
\]
and finally the equilibrium equations (59) can be written
\[
-N^a_0 g_{\mu \beta}, x] + N^a_0 \Gamma^a_{a_j} g_{\mu \beta} = p^0_n
\]
\[
N^a_0 C^a_{a_j} = -p^0_n.
\]
(65)
These equations can be obtained directly under an intrinsic formulation by pulling back differently the Eulerian membrane equations so as to separate the normal from the tangential part. This other pull-back avoids using the non-classical divergence \( \text{div}_{\mu} \) but is more complex. The complete calculus are presented in Appendix A. □

8. Back to dimensional variables

To finish it, is possible to return to the physical variables in the Eulerian membrane equations of Proposition 3. To do this let us define
\[
\varphi^* = \phi^* \varphi^0 = L_0 \varphi^0.
\]
We then have the following proposition:

Proposition 7. For a severe level of applied forces \( f^* \) and \( g^* \) such as \( \varphi^* = \varphi^0 = T^2 \) and \( \varphi^* = \varphi^0 = T^2 \), \( \varphi^* \) depends only on \( p^* \) and is solution of the following non-linear membrane problem:
\[
h_0 \text{div}^*(n^*_{i}) = -\bar{p}_i \text{ in } \omega^*;
\]
\[
h_0 \text{Tr}(n^*_{i} C^*) = -p^* \text{ in } \omega^*,
\]
\[
\varphi^* = i_{x^*} \text{ on } \gamma_1^*,
\]
\[
n^*_{i} v^* = 0 \text{ on } \gamma_2^*
\]
(66)
with
\[
n^*_{i} = \frac{4 \lambda \mu}{\lambda + 2 \mu} \text{ Tr } A^i_{0} I_{T^0} + 4 \mu A^i_{0},
\]
\[
A^i_{0} = \frac{1}{2} \left( I_{T^0} - \frac{\partial \varphi^*}{\partial p^*} \frac{\partial \varphi^*}{\partial p^*} \right),
\]
\[
p^*_i = g^*_i + g^*_i \text{ and } p^*_n = g^*_n + g^*_n.
\]

The proof of this proposition is evident. Indeed, going back to the dimensional variables in equations of Proposition 3, we get
\[
L_0 \varphi^* \text{div}^*(n^*_{i}) = -\bar{p}_i^*,
\]
\[
\frac{G_n}{C_r} \text{ Tr}(n^*_{i} C^*) = -p^*_n.
\]
According to the level of surface forces considered and to the order of magnitude of \( h_0 C_r = \nu \), we obtain Eqs. (66).
9. Comparison with equations obtained with a surfacic approach

With direct surfacic approaches [22,23,26–28], which modelize the shell as a two-dimensional surface embedded in $\mathbb{R}^3$, it is equally possible to obtain general equilibrium equations of shells. These equilibrium equations obtained without any assumption are valid for large strains. The difficulty is then to obtain a surfacic constitutive law, between static and kinematic variables. To do this, we generally have recourse to a priori assumptions, and the rigorous character of surfacic approaches is lost.

We recall the general Eulerian equations of membrane $^{12}$ naturally obtained by Breuneval [27,28] in the deformed configuration with a surfacic approach

\[
\begin{align*}
L_0 \text{div}^s N_i^* &= -\hat{F}_i^s \quad \text{in } \omega^s, \\
L_0 \text{Tr}(N_i^* C^s) &= -\hat{F}_n^s \quad \text{in } \omega^s, \\
N_i^* v^* &= f^* \quad \text{on } \gamma^s, \\
\end{align*}
\]

where $\hat{F}_i^s$, $\hat{F}_n^s$ and $f^*$ denote the resultant of the Eulerian forces acting in the deformed configuration $\omega^s$, $N_i^*$ the superficial stress tensor and $v^*$ the unit external normal to $\gamma^s = \partial \omega^s$. The physical interpretation of $N_i^*$, naturally introduced by duality, and the obtaining of a simple surfacic constitutive law which links $N_i^*$ to the membrane strain tensor, needs to have recourse to a priori assumptions.

Within the framework of the classical Kirchhoff-Love theory, $^{13}$ for a tangentially isotropic membrane subjected to small strains, it can be proved $^{14}$ that the linearized expression of $N_i^*$ coincides with $n_i^{*0}$ defined in Proposition 7. In this case, the Eulerian equations of membrane of Proposition 7 are identical to those obtained with a direct surfacic approach. However, contrary to surfacic approaches which require to introduce assumptions on the constitutive law, the asymptotic Eulerian approach developed in this paper enables to obtain a non-linear Eulerian membrane model, valid for large displacements, without any a priori assumption.

10. Conclusion

For a shell subjected to severe forces, the asymptotic Eulerian approach developed in this paper enables to construct a non-linear membrane model which takes into account the “real forces” acting in the deformed configuration. To our knowledge, there is no equivalent in the literature where the asymptotic approaches, using a Lagrangian description of the three-dimensional problem, generally consider dead loads.

The intrinsic formalism used enables to pull-back the Eulerian membrane equations obtained in the reference configuration—or in any intermediate one—to Lagrangian variables. This pull-back takes into account the direction of the applied forces as well as enables to consider following forces.

Thus, the approach developed in this paper is particularly well adapted for fluid–structure interaction problems. Indeed, on one hand, the Eulerian model of membrane obtained is directly usable in an Eulerian formulation of the problem for both the fluid and the structure, as in [12]. On the other hand, the pull-back of the Eulerian membrane model can be carried out to any Lagrangian configuration. It is then equally possible to use an A.L.E formulation of the fluid–structure interaction problem [9].

Appendix A. Classical pull-back of Eulerian membrane equation

We present in this appendix another way to pull-back the Eulerian membrane equations of Proposition 3 without using the divergence $\text{div}_{\omega^s}$. To do this we have to pull-back separately the in plane and the normal components of Eq. (26), once written into the following equivalent weak formulation:

\[
\varphi_0 \in \mathcal{A}(\omega) = \{ \phi : \omega \to \mathbb{R}^3, \text{“smooth”, } \phi = i_\omega \text{ on } \gamma_1 \},
\]

(A.1)
\[
\int_\omega \text{Tr}\left( n_0^\dagger \frac{\partial \psi_t}{\partial p} \right) \, d\omega = \int_\omega \tilde{\mathcal{P}}_t v_t \, d\omega \quad \forall v_t \in V_T(\omega) = \{ v_t : \omega \rightarrow T_p \omega, \text{“smooth”}, v_t = 0 \text{ on } \gamma_1 \},
\]

\[
- \int_\omega \text{Tr}(n_0^\dagger C) v_n \, d\omega = \int_\omega p_n v_n \, d\omega \quad \forall v_n \in V_n(\omega) = \{ v_n : \omega \rightarrow \mathbb{R}, \text{“smooth”}, v_n = 0 \text{ on } \gamma_1 \}. \tag{A.2}
\]

Let us define now the space of admissible tangent and normal displacements on \( \omega_0 \):

\[
V_T(\omega_0) = \{ v_0^0 : \omega_0 \rightarrow T_p \omega_0, \text{“smooth”}, v_t^0 = 0 \text{ on } \gamma_{01} \},
\]

\[
V_n(\omega_0) = \{ v_n^0 : \omega_0 \rightarrow \mathbb{R}, \text{“smooth”}, v_n^0 = 0 \text{ on } \gamma_{01} \}.
\]

We then have the following proposition:

**Proposition 8.** The pull-back on \( \omega_0 \) of Eqs. (A.1) and (A.2) leads to the following Lagrangian weak formulation:

\[
\psi^0 \in \mathcal{A}(\omega_0) = \{ \psi : \omega_0 \rightarrow \mathbb{R}^3, \text{“smooth”}, \psi = i_{\omega_0} \text{ on } \gamma_0 \}
\]

\[
- \int_{\omega_0} \text{div}(N_0^0 \mathcal{G}) v_0^0 \, d\omega_0 + \int_{\omega_0} \text{Tr}(N_0^0 \mathcal{V} N_0^0) \, d\omega_0
\]

\[
= \int_{\omega_0} \tilde{\mathcal{P}}^0_0 v_0^0 \, d\omega_0 - \int_{\gamma_{02}} \tilde{v}_0^0 \mathcal{D}(v_0^0) \, d\gamma_0
\]

\[
\forall v_0^0 \in V_T(\omega_0), \tag{A.3}
\]

\[
\int_{\omega_0} \text{Tr}(N_0^0 \tilde{\mathcal{C}}) v_n^0 \, d\omega_0 = - \int_{\omega_0} P_n^0 v_n^0 \, d\omega_0
\]

\[
\forall v_n^0 \in V_n(\omega_0), \tag{A.4}
\]

with

\[
\tilde{\mathcal{C}} = \frac{\partial \mathcal{P}}{\partial p} \tilde{C} \frac{\partial \mathcal{P}}{\partial p}, \quad \mathcal{G} = \frac{\partial \mathcal{P}}{\partial p} \mathcal{P} \frac{\partial \mathcal{P}}{\partial p},
\]

\[
P_t^0 = \sqrt{g} \frac{\partial \mathcal{P}}{\partial p} p_t \circ \psi^0 \quad \text{and} \quad P_n^0 = \sqrt{g} p_n \circ \psi^0,
\]

where \( \mathcal{D}(v_0^0) \) denotes the Riemannian connexion of the mapping \( p_0 \mapsto p = \psi^0(p_0) \) as follows:

\[
\mathcal{D}(v_0^0) = \left( \frac{\partial \psi_t}{\partial p} \right)^{-1} \frac{\partial \psi_t}{\partial p} - \frac{\partial \psi_t}{\partial p}.
\]

\( \mathcal{D}(v_0^0) \) represents the difference of the connections \( \Gamma - \Gamma_0(Y) \) where \( Y \) denotes the components of the vector fields \( v_t = SY \) and \( v_n^0 = S_0 Y \) in local coordinates. Going back to \( \omega_0 \), Eq. (A.1) becomes

\[
\int_{\omega_0} \text{Tr}\left( N_0^0 \frac{\partial \mathcal{P}}{\partial p} \frac{\partial \mathcal{P}}{\partial p} \left[ \frac{\partial \psi_t}{\partial p} + \mathcal{D}(v_0^0) \right] \right) \, d\omega_0
\]

\[
= \int_{\omega_0} \tilde{\mathcal{P}}_t^0 v_t^0 \, d\omega_0 \quad \forall v_t^0 \in V_T(\omega_0),
\]

where \( \tilde{\mathcal{P}}_t^0 = \sqrt{g}(\frac{\partial \mathcal{P}}{\partial p} p_t \circ \psi^0 \) denotes the pull-back of the tangential forces.\(^{15}\) According to the definition

\(^{15}\)Let us remark that this pull back keeps the direction of the forces. Indeed the forces \( \tilde{P}_t^0 \) are tangent to \( \omega_0 \) because of the term \( \frac{\partial \mathcal{P}}{\partial p} p_t \).
of $G = (\partial p/\partial p_0)(\partial p/\partial p_0)$ (see [18,27]), we get

$$
\int_{\omega_0} \text{Tr} \left( N_i^0 G \frac{\partial v^0_i}{\partial p_0} \right) \, d\omega_0 + \int_{\omega_0} \text{Tr} (N_i^0 G \mathcal{D}(v^0_i)) \, d\omega_0
$$

$$
= \int_{\omega_0} \mathcal{P}^0_i v^0_i \, d\omega_0 \quad \forall v^0_i \in V_T(\omega_0).
$$

As $v^0_i = 0$ on $\gamma_{01}$, we obtain finally the first equation of Proposition 8 where $\text{div}$ denotes the divergence on $\omega_0$. □

### A.1. Particular case of plates

Equations of Proposition 8 are a priori equivalent to the Lagrangian equations of Proposition 5 obtained differently. To prove this, let us write Eqs. (A.3) and (A.4) of Proposition 8 in local coordinates in the particular case of plates, where $(\omega_0, \psi^0)$ can be considered as the system of coordinates of $\omega$. We will see that the equations so obtained coincide with those of Proposition 6.

For a plate, the expression of $G$ reduces to $G$, the classical first fundamental form of $\omega$. We note $g_{2\psi}$ the components of $G$ in the system of local coordinates.

On the other hand, let $Y = Y^i e_1 + Y^2 e_2$ be the components of $v^0_i$ in the Cartesian basis $(e_1, e_2)$ related to the system of local coordinates. The Riemannian connexion $\mathcal{D}(v^0_i)$ of the mapping $p_0 \mapsto p$ then reduces to $\mathcal{D}(v^0_i) = \Gamma(Y)$. In local coordinates, its components are given by

$$\mathcal{D}(v^0_i) = \Gamma^i_{2\mu} Y^\mu$$

where $\Gamma^i_{2\mu}$ denotes the Christoffel symbol of the surface $\omega$. Thus the second term of (A.3) can be written in local coordinates

$$\text{Tr}(N_i^0 G \mathcal{D}(v^0_i)) = N^0_{2g} g_{2\rho} \mathcal{D}^i_{2\rho} = N^0_{2g} g_{2\rho} \Gamma^i_{2\mu} Y^\mu,$$

where $N_i^0 G$ and $\mathcal{D}(v^0_i)$ are two endomorphisms of $\mathbb{R}^2$. In the same way the first term of (A.3) is given by

$$\text{div}(N_i^0 G)v^0_i = [N^0_{2g} g_{2\rho}]_{,x} Y^\mu.$$