Limit behavior of Koiter model for long cylindrical shells and Vlassov model

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ABSTRACT

We propose in this article to consider the limit behavior of the Koiter shell model when one of the characteristic length of the middle surface becomes very large with respect to the other. To do this, we perform a dimensional analysis of Koiter formulation which involves dimensionless numbers characterizing the geometry and the loading. Once reduced to a one-scale problem corresponding to thin-walled beams (long cylindrical shell), using asymptotic expansion technique, we address the limit behavior of Koiter model when the aspect ratio of the shell tends to zero. We prove that at the leading order, Koiter shell model degenerates to a one dimensional thin-walled beam model corresponding to the Vlassov one. Moreover, we obtain a general analytical expression of the geometric constants involved, that improves the empirical expression given by Vlassov.

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1. Introduction

Thin structures (plates, shells, beams and thin-walled beams) are widely used in industries because they provide a maximum of stiffness with a minimum of weight. Among thin-structures, we classically distinguish plates (with zero curvature), shells (shallow and curved shells), beams and thin-walled beams. Thin-walled beams are at the cross of shells and classical beams (with full cross-section): they can be seen equivalently as beams whose profile of a cross-section is thin, or as very long shells.

Classical one-dimensional or two-dimensional models of plates, shells, beams and thin-walled beams were obtained historically using a priori kinematics and statics assumptions in the three-dimensional equilibrium equations (Koiter, 1960; Novozhilov, 1959; Vlassov, 1962). More recently, asymptotic approaches 1 enabled to justify rigorously most of these classical models (Carlet and Destuynder, 1979; Destuynder, 1985; Sanchez-Palencia, 1989a,b, 1990; Hamdouni and Millet, 2003a,b; Rigolot, 1977; Marigo et al., 1998. However, in spite of all these works, the Koitershell model was never justified by asymptotic approach 2 although it is one of the most used for computing linear elastic shell problems.

A general feature of these asymptotic approaches (according to boundary conditions), is that in the asymptotic behavior of the variational displacement approach, penalty terms naturally emerge, leading to a limit problem in a constrained sub-space. Generally, the sub-space so defined by the penalty terms, corresponds to the searched classical kinematics (Kirchhoff-Love in plate theory, Novozhilov–Donnell for shallow shells, or pure bendings for geometrically non-rigid shells).

The reference model for thin-walled beams in linear elasticity is the Vlassov model 3 (Vlassov, 1962), historically established using a priori kinematics and statics assumptions. Similar a priori assumptions are used in recent works dealing with thin walled beam theory and applications (Kim and Kim (2005), Bottino et al. (2005), El Fatmi (2007)).

In the literature, there exists only a few works on the rigorous justification of Vlassov model using asymptotic methods. First results were obtained in (Rodriguez and Víaño, 1995, Rodriguez and Víaño, 1997, Trabucho and Víaño, 1996), using an approach based on an expansion at the second order with respect to the diameter of the beam, to obtain an enriched model. Then in a second time, the thickness is assumed to tend to zero: that leads to a thin-walled beam model similar to Vlassov one.However, it is well-known that these two operations do not commute and the result depends on the choice made. 4

Afterwards, a justification of Vlassov model from the asymptotic expansion of the three-dimensional elasticity equations was 5 which is a one-dimensional model composed of four differential equations.

4 This is a classical result in homogenization of periodic structures (Caillerie, 1984, Lewinski, 1991).
proposed in (Grillet et al., 2000, Hamdouni and Millet, in press). In this work, the width and the thickness of the profile tend together to zero. This way, for thin-walled beams with open strongly bent cross-section, the asymptotic model obtained differs slightly from Vlassov one: a supplementary term coupling bending and twist remains in the bending reduced equations. In the same manner, an asymptotic thin-walled beam model was obtained for shallow profiles (Grillet et al., 2005), and an extension to the non-linear case was proposed in (Grillet et al., 2004). In (Volovoi and Hodges, 2000), the authors proposed an anisotropic thin-walled beam model obtained from the Koiter model\(^5\) using the variational-asymptotic method. Finally, let us also notice the works of Diaz and Sanchez-Palencia (2007), where convergence results to a thin-walled beam model for shallow profiles was established starting from Novozhilov–Donnell model.

In this paper, we address the limit behavior of Koiter shell model when one of the characteristic dimension of the middle surface becomes much larger than the other. We prove in particular that the Koiter model degenerates to a one dimensional thin-walled beam model corresponding to Vlassov model, when the aspect ratio tends to zero. Moreover, we obtain a general analytical expression of the geometric constants involved, that improves the empirical expression given by (Vlassov, 1962).

The paper is organized as follows. Section 2 is devoted to the formulation of the problem and of the linear Koiter shell model for a cylindrical shell with open cross-section. Then in Sections 3 and 4, we perform a dimensional analysis of Koiter formulation that makes naturally appear dimensional numbers characterizing the geometry and the loading. Once reduced to a one-scale problem corresponding to thin-walled beams (or to long cylindrical shells), using asymptotic expansion technique, we examine the limit behavior of the variational Koiter model when the aspect ratio of the shell tends to zero.\(^6\) We shall see that several penalty terms appears in the weak formulation of the Koiter model, leading to the classical Vlassov kinematics for thin-walled beams. Thus, without any a priori assumption, the leading term of the asymptotic expansion of the displacements is proved to verify the classical Vlassov kinematics. Then, in Section 5, we prove that the constrained Koiter problem posed in this sub-space of Vlassov kinematics, degenerates to four ordinary differential equations (in the variable of the mean fiber of the beam) characterizing the three components of the displacement, and the twist angle. The limit one-dimensional beam model so obtained for very long shells in then compared to Vlassov equilibrium equations, and to the results obtained in (Grillet, 2003; Hamdouni and Millet, in press) from the asymptotic expansion of the three-dimensional equations of linear elasticity.

2. Description of the problem

2.1. Geometry and parametrization

In all that follows, the dimensional variables will be denoted with a tilde, whereas the dimensionless one will be denoted without. Moreover, the over-lined functions will denote functions which depend only on the variable \(\tilde{y}_1\), associated to the longitudinal direction of the generators of the cylindrical shell (Fig. 1).

Let us consider a linear elastic thin-walled beam with open cross-section, or equivalently a long cylindrical shell (Fig. 1). It is described by the open cylindrical middle surface \(S\) and the constant thickness \(2h\). In this paper, we limit our study to thin-walled beams with open strongly curved profile, whose curvature is denoted \(\tilde{c}\) and length \(d\). The length of the beam is denoted \(L\) and is assumed to be much longer than the length \(d\) of the profile.

Let us consider a cartesian coordinate system \((O, e_1, e_2, e_3)\) associated with the three-dimensional space, and a local coordinate system \((\tilde{p}, a_1, a_2, N)\) associated with each point \(p\) of the profile, where \(a_1 = e_1\) is the direction of the generators (see Fig. 1). In the case of cylindrical shells, we can always consider a mapping \(\mathcal{P}(\tilde{y}_1, \tilde{y}_2)\) of the middle surface \(S\) such as \((a_1, a_2, N)\) is orthonormal, where \((\tilde{y}_1, \tilde{y}_2) \in \Omega\) denotes the local variables associated with the parametrization. Consequently, we have \(a_{1p} = \delta_{1p}\), and the contravariant basis \((a^1, a^2, N)\) confuses with the covariant basis \((a_1, a_2, N)\). Therefore, in the next, we will use indifferently the covariant or the contravariant components of the considered vector and tensor fields. Vectors will be denoted \(\tilde{u} = U e_1\) in the cartesian basis and \(\tilde{u} = u_i a_i\) in the local basis, with \(a_3 = N\). Moreover, the coordinates of a point \(\tilde{p} = \mathcal{P}(\tilde{y}_1, \tilde{y}_2)\) will be denoted \((x_1, x_2, x_3)\) in the cartesian basis.

We assume that the shell considered is subjected to forces with surface densities \(f = (f_1, f_2, f_3)\), and is fixed or clamped at its two extremities \(\tilde{F}_0\) and \(\tilde{F}_1\) corresponding respectively to \(\tilde{y}_1 = 0\) and \(\tilde{y}_1 = L\). Moreover, the lateral boundary \(\tilde{F}_1\) is free.

2.2. The Koiter shell model

We consider in this paper long cylindrical shells in linear elasticity, whose mechanical behavior is described by the Koiter shell model. Its variational formulation for a shell with a thickness \(2h\) subjected to a surface loading \(f\) classically writes (Destuynder, 1985; Bernadou, 1996; Sanchez-Hubert and Sanchez-Palencia, 1997; Béchet et al., 2008):

\[2h \int_S A^{ijkl} \gamma_{ij}(\tilde{u}) \gamma_{kl}(\tilde{u}) dS + \int_S A^{ijkl} \hat{p}_{ij}(\tilde{u}) \hat{p}_{kl}(\tilde{u}) dS = 2h \int_S \tilde{f} \tilde{v} dS \quad \forall \tilde{v} \in \tilde{V}\]

\[(1)\]

\[\tilde{u} \in \tilde{V} = \{H^1(\tilde{\Omega}) \times H^1(\tilde{\Omega}) \times H^2(\tilde{\Omega})\}, \quad \tilde{v} \text{ satisfying the kinematics boundary conditions, such that:}\]

\[\text{Fig. 1. The thin-walled beam and the used coordinate systems.}\]
where
\[ A^{\alpha\beta\mu} = \frac{E}{2(1+\nu)} \left[ \delta^{\alpha\beta} \delta^{\mu\mu} + \delta^{\alpha\mu} \delta^{\beta\beta} + \frac{2\nu}{1-\nu} \delta^{\beta\mu} \delta^{\alpha\beta} \right] \]  
(2)

denote the coefficients of the elastic constitutive law. In particular, we have:
\[ \begin{align*}
A^{1111} &= A^{2222} = \frac{4\mu(\nu+1)}{2(1+\nu)} \\
A^{1122} &= A^{2211} = \frac{2\mu}{2\nu+3} \\
A^{1212} &= A^{2112} = \mu \\
A^{1211} &= A^{2122} = 0
\end{align*} \]  
(3)

where we set \( \gamma = \frac{2}{3} \) and where \( \lambda \) and \( \mu \) are the Lamé coefficients of the material considered. In the next, to simplify the notations, we set \( A = \frac{E}{1+\nu} \) and \( B = \frac{4\mu}{3(1-\nu)} \).

On the other hand, the components \( \gamma_{xy} \) and \( \rho_{xy} \) of the membrane strain tensor and of the variation of curvature tensor are given by:
\[ \gamma_{xy}(\bar{u}) = \frac{1}{2} (\bar{D}_x \bar{u}_y + \bar{D}_y \bar{u}_x) - \bar{b}_{xy} \bar{u}_3 \]  
(4)

and
\[ \rho_{xy}(\bar{u}) = \bar{\partial}_x \bar{\partial}_y \bar{u}_3 - \bar{\Gamma}_{xy} \bar{\partial}_x \bar{u}_3 - \bar{b}_{y}^2 \bar{u}_3 + \bar{D}_x (\bar{b}_y^2 \bar{u}_1) + \bar{b}_y^2 \bar{D}_x \bar{u}_1 \]  
(5)

where
\[ \bar{D}_x \bar{u}_y = \bar{\partial}_x \bar{u}_y - \bar{\Gamma}_{xy} \bar{u}_3 \]  
(6)

denotes the covariant derivative of \( \bar{u}_3 \), \( \bar{b}_{xy} \) being the coefficients of the second fundamental form accounting for curvatures. Finally, \( \bar{\partial}_x \) is the classical derivative with respect to \( x \) and \( \bar{\Gamma}_{xy} \) denote the Christoffel symbols of the middle surface \( \bar{S} \).

For the cylindrical shells considered in this paper, only one component of the curvature tensor \( b_{22} = b_2 = \bar{c} \) does not vanish when considering the parametrization of Fig. 1. Moreover the curvature only depends on \( y_2 \). Consequently, the components (4) and (5) of the membrane strain tensor and of the variation of curvature tensor reduce to:
\[ \begin{align*}
\gamma_{11} &= \bar{u}_{1,1} \\
\gamma_{22} &= 0 \\
\gamma_{12} &= \frac{1}{2} (\bar{u}_{1,2} + \bar{u}_{2,1}) \\
\gamma_{11} &= \bar{u}_{1,1} \\
\gamma_{22} &= \bar{u}_{2,2} - \bar{c} \bar{u}_3 \\
\gamma_{12} &= \frac{1}{2} (\bar{u}_{1,2} + \bar{u}_{2,1}) \\
\rho_{11} &= \bar{u}_{3,1} \\
\rho_{22} &= \bar{u}_{3,2} + \bar{c} \bar{u}_2 \\
\rho_{12} &= \bar{u}_{3,2} + \bar{c} \bar{u}_2 \\
\rho_{12} &= \bar{u}_{3,2} + \bar{c} \bar{u}_2
\end{align*} \]  
(7)

Note that it would be also possible to start from Novozhilov equations\(^7\) for cylindrical shells (see 40 of Novozhilov, 1959). However, as the Novozhilov equations are similar (but not identical) to Koiter formulation for cylindrical shells (the bending contribution is simplified in Novozhilov equations), we would obtain the same Vlassov kinematics of Result 4.1, but not the one-dimensional equilibrium equations of Section 6.

3. Dimensional analysis of equations

3.1. Scaling of the geometry, forces and unknowns

First, let us define the following dimensionless data and unknowns of the problem (noted without a tilde), like in Grillet,
\[ y_1 = \frac{y_1}{L}, \quad y_2 = \frac{y_2}{d}, \quad \bar{c} = \frac{c}{\bar{c}}, \quad f_1 = \frac{f_1}{f_2}, \quad f_2 = \frac{f_2}{f_3}, \quad f_3 = \frac{f_3}{f_3} \]  
(9)

The variables indexed with \( r \) are the reference ones, for the applied forces \( (f_{1r}, f_{2r}, f_{3r}) \), the unknown displacements \( (u_{1r}, u_{2r}, u_{3r}) \) or the geometry \( (d, L) \). Consequently, we have \( \partial_1 = \frac{1}{d} \partial_1 \) and \( \partial_2 = \frac{1}{d} \partial_2 \), where \( \partial_3 = \frac{1}{d} \partial_3 \).

In what follows, we consider that the reference curvature \( c_r \) is equal to \( \bar{c}_{\text{max}} \), the maximum of \( |\bar{c}| \), where \( |\bar{c}| \) denotes the absolute value of \( \bar{c} \). This is equivalent to assume that the curvature stays of the same order of magnitude along the whole profile. In this paper, we focus on thin-walled beams with a strongly curved profile such as \( 1/c_r \approx d \).

Moreover, to avoid any \( a \) priori assumption on the order of magnitude of the displacements (which are the unknowns of the problem), we consider in a first time that \( u_{1r} = u_{2r} = u_{3r} = h \), which ensures to stay in the framework of linear elasticity. Then, using (9) and (10), we obtain the following expressions of the dimensionless components of the membrane strain tensor:
\[ \begin{align*}
\bar{\gamma}_{11} &= \frac{u_{1,1}}{L} \\
\bar{\gamma}_{22} &= \frac{u_{2,2}}{d} - \bar{c} \bar{u}_3 \\
\bar{\gamma}_{12} &= \frac{1}{2} (\bar{u}_{1,2} + \bar{u}_{2,1}) \\
\bar{\rho}_{11} &= \frac{u_{3,1}}{d} \\
\bar{\rho}_{22} &= \frac{u_{3,2}}{d} + \bar{c} \bar{u}_2 + (\bar{c} \bar{u}_2)_2 - \bar{c}^2 \bar{u}_3 \\
\bar{\rho}_{12} &= \frac{u_{3,2}}{d} + \bar{c} \bar{u}_2
\end{align*} \]  
(11)

and of the curvature variation tensor:
\[ \begin{align*}
\bar{\rho}_{11} &= \frac{u_{3,1}}{d} \\
\bar{\rho}_{22} &= \frac{u_{3,2}}{d} + \frac{\bar{c}}{d} [u_{2,2} + (u_{2,2})_2] - \frac{1}{d^2} \bar{c}^2 u_{3,3} \\
\bar{\rho}_{12} &= \frac{u_{3,2}}{d} + \frac{\bar{c}}{d} [u_{2,2} + (u_{2,2})_2] - \frac{1}{d^2} \bar{c}^2 u_{3,3}
\end{align*} \]  
(12)

where we set \( \bar{\varepsilon} = \frac{2}{d}, \quad \bar{\eta} = \frac{h}{d} \) and \( \bar{\nu} = \frac{h}{c_r} \).

Thus, three dimensionless numbers characterizing the geometry of thin-walled beams or of long cylindrical shells naturally emerge:

- \( \bar{\varepsilon} = \frac{2}{d} \) denotes the aspect ratio of the beam. It is a small parameter.
- \( \bar{\eta} = \frac{h}{d} \) represents the relative thickness of the beam (compared to the length of the profile). It is also a small parameter for thin-walled beams.
- \( \bar{\nu} = \frac{h}{c_r} \) is the ratio between the thickness and the smallest radius of curvature of the profile. Equivalently, it represent the shallowness of the corresponding shell.

Therefore, three dimensionless geometric numbers are necessary to describe accurately the geometry of thin-walled beams, whereas only two are needed for shells.

On the other hand, the dimensional analysis of the right-hand side of (1) leads to:
\[ \int_2^3 \bar{f} \cdot \bar{v} dS = \int_2^3 (f_1 \bar{v}_1 + f_2 \bar{v}_2 + f_3 \bar{v}_3) dS = \mu L dh \int (G_1 \bar{f}_1 v_1 + (G_2) \bar{f}_2 v_2 + (G_3) \bar{f}_3 v_3) dS \]  
(13)

where we set:
\[ G_1 = \frac{f_1}{\mu}, \quad G_2 = \frac{f_2}{\mu}, \quad G_3 = \frac{f_3}{\mu} \]  
(14)
The Lamé coefficient $\mu$ is considered as a reference stress. These dimensionless numbers characterize the order of magnitude of the applied forces.

### 3.2. One scale problem

In order to get a one-scale problem to perform the asymptotic expansion of dimensionless Koiter model, we chose $\varepsilon$ as the reference small parameter, and we express the other dimensionless numbers with respect to the powers of $\varepsilon$. First, in this paper, we will only consider thin-walled beams with strongly curved profile such that:

$$\eta = \varepsilon \quad \text{and} \quad \nu = \varepsilon$$

(15)

Note that the condition $\nu = \varepsilon$ implies that $1/\varepsilon \approx d$, which corresponds to strongly curved profiles, for instance a half-cylinder. Moreover, we will consider moderate levels of applied forces, of the same order of magnitude as in Grillet, 2003, in press:

$$G_1 = \varepsilon^5, \quad G_2 = \varepsilon^6, \quad G_3 = \varepsilon^8$$

(16)

For these levels of applied forces, we can prove (with a development similar to Handmou and Millet, in press) that the longitudinal displacement $u_1$ is one order smaller than the two bending displacements. This result expresses the fact that the traction displacement is smaller than bending displacements for this kind of structure, whose traction rigidity is more important than the bending one. Thus, we finally consider the following scalings for the displacements:

$$u_{1r} = zh, \quad u_{2r} = h, \quad u_{3r} = h$$

(17)

Using scalings (15) and (17), we obtain a problem which depends only on the small perturbation parameter $\varepsilon$, where the expressions of $\tilde{\gamma}_{xy}$ and $\rho_{xy}$ reduce to:

$$\tilde{\gamma}_{11} = \varepsilon^3 \gamma_{11}$$

$$\tilde{\gamma}_{22} = \varepsilon^2 \gamma_{22}$$

$$\tilde{\gamma}_{12} = \varepsilon \gamma_{12}$$

with

$$\gamma_{11} = u_{1,1}$$

$$\gamma_{22} = u_{2,2} - cu_3$$

$$\gamma_{12} = \frac{1}{2} (u_{1,2} + u_{2,1})$$

(18)

Replacing expressions (18) of $\tilde{\gamma}_{xy}$ and $\rho_{xy}$ in the dimensionless formulation of Koiter model (1), we obtain a new dimensionless variational formulation which writes on the form of the following steps:

- First, in this paper, we will only consider thin-walled beams with strongly curved profile $\varepsilon$ such that:

$$u_1 = u_1^0 + \varepsilon u_1^1 + \varepsilon^2 u_1^2 + \cdots$$

$$u_2 = u_2^0 + \varepsilon u_2^1 + \varepsilon^2 u_2^2 + \cdots$$

$$u_3 = u_3^0 + \varepsilon u_3^1 + \varepsilon^2 u_3^2 + \cdots$$

(21)

Let us now replace the expressions (21) of the displacements in (20). We obtain a chain of coupled problems $\varepsilon^{-4} \leq \varepsilon^6$, corresponding to the cancellation of the factors of the successive powers of $\varepsilon^{-4}$ to $\varepsilon^6$. The calculations that follow will be performed in two main steps:

- first the penalty terms (corresponding to $\varepsilon^6$ with $p < 0$) will exhibit the specific Vlassov kinematics$^9$ satisfied by the leading term $(u_1^0, u_2^0, u_3^0)$ of the expansion (21),
- then problem $\varepsilon^0$ at the leading order $\varepsilon^0$ will characterize the specific one-dimensional equilibrium equations of the problem (the search one-dimensional thin-walled beam model).

### Result 4.1. For applied force levels such as $G_2 = G_3 = \varepsilon^6$ and $G_1 = \varepsilon^8$, the leading term $u^0 = (u_1^0, u_2^0, u_3^0)$ of the expansion of the displacement satisfies the Vlassov kinematics which writes:

$$u_1^0 = U_1 - x_2 \frac{dU_1}{dy_1} - x_3 \frac{dU_1}{dy_1} - \omega \frac{d\Theta_1}{dy_1}$$

$$u_2^0 = U_2 \cos \alpha + U_3 \sin \alpha - q(y_2) \Theta_1$$

$$u_3^0 = -U_2 \sin \alpha + U_3 \cos \alpha + l(y_2) \Theta_1$$

with

$$l(y_2) = (x_2 - x_2^0) \cos \alpha + (x_3 - x_3^0) \sin \alpha$$

$$q(y_2) = -(x_2 - x_2^0) \sin \alpha + (x_3 - x_3^0) \cos \alpha$$

$$\frac{d\alpha}{dy_2} = c$$

(22)

and where$^{10}$

- $U_1$, is the longitudinal displacement in the direction $e_1 = a_1$.
- $U_2$, and $U_3$, are the cartesian displacements in $e_2$ and $e_3$ direction of an arbitrary point $C$ of coordinates $(x_2^0, x_3^0)$ in the plane of a cross-section.
- $\Theta_1$ denotes the rotation of the profile around the axis $(C, e_1)$.
- $\omega$ corresponds to the sectorial area defined by:

$$\frac{d\omega}{dy_2} = -q$$

$^9$ Of rigid solid in the plane of a cross-section.

$^{10}$ We recall that the over-lined functions denote functions which depend only on the variable $y_1$. 

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In such asymptotic calculations, expression (20) contains terms with factors $\varepsilon^p$, $p \in \mathbb{N}$. The terms with factors $p > 0$ are “penalty terms” (restraining the space $V$ to a subspace $G$ as we will see in the sequel), whereas those with $p > 0$ are called “singular perturbation terms”, which are lost at the limit. Finally, terms with factor $\varepsilon^0$ contain the limit problem at the main order, which is generally well-posed in the subspace $G$ accounting the penalty conditions.

### 4. Vlassov kinematics

According to the classical asymptotic technique Sanchez-Hubert and Sanchez-Palencia, 1992, we search for the solution $u = (u_1, u_2, u_3)$ under the form of a formal expansion with respect to the perturbation parameter $\varepsilon$:

[Note: The rest of the text contains more detailed mathematical expressions and derivations related to the Vlassov kinematics, including derivations of the Vlassov equations and boundary conditions, but these are not shown here for brevity.]
Proof. The proof of this result is split into four steps from (i) to (iv).

Result 4.1 provides directly from the penalty terms associated with the singular perturbation problem (20). They correspond to the cancellation of the terms with factors $\varepsilon^{-4}$ and $\varepsilon^{-2}$. In the same way, the cancellation of the terms with factors $\varepsilon^{-3}$ and $\varepsilon^{-1}$ lead exactly to the same results for the second term $u^1 = (u^1_1, u^1_2, u^1_3)$ of the expansion (21).

(i) Characterization of the subspace $G$

The cancellation of the factor of $\varepsilon^{-4}$ in (20) leads to the penalty condition $P^{-4}$ which writes:

### Find $u^0 \in V$ such as $\int_S A \gamma_{22} (u^0)^2 \, dS = 0 \quad \forall \nu \in V$ (24)

Considering the particular test displacement $\nu = u^0 \in V$, we get:

### $\int_S A \gamma_{22} (u^0)^2 \, dS = 0$ (25)

As $A$ is a strictly positive constant, $\gamma_{22} (u^0)$ vanishes so that we obtain according to (19):

### $\gamma_{22} (u^0) = 0 \iff \partial_2 u^0_2 - cu^0_3 = 0$ (26)

Now, the cancellation of the factor of $\varepsilon^{-2}$ leads to the penalty condition $P^{-2}$ which writes:

### $\int_S [B \gamma_{22} (u^0)^2 \gamma_{11} (\nu) + 8 \gamma_{12} (u^0) \gamma_{12} (\nu) + A \gamma_{22} (u^2)^2 \gamma_{22} (\nu)] + B \gamma_{11} (u^0)^2 \gamma_{22} (\nu) \, dS = 0 \quad \forall \nu \in V$ (27)

Since $\gamma_{22} (u^0) = 0$ according to (26), taking $\nu = u^0$ again, Eq. (27) reduces to:

### $\int_S 8 [\gamma_{12} (u^0)^2]^2 \, dS + \int_S A \rho_{22} (u^0)^2 \, dS = 0$ (28)

which implies that:

### $\gamma_{12} (u^0) = 0 \iff \partial_2 u^0_2 + \partial_1 u^0_3 = 0$ (29)

### $\rho_{22} (u^0) = 0 \iff \partial_2 u^0_2 + \partial_2 u^0_3 + \partial_3 (cu^0_2) - c^2 u^0_3 = 0$ (30)

Thus, we define the subspace $G \subset V$ incorporating (26) and (29) as constraints on the displacement associated with the singular perturbation problem (20):

### $G = \{u^0 \in V, \text{ such that } \gamma_{22} (u^0) = 0, \rho_{22} (u^0) = 0, \gamma_{12} (u^0) = 0\}$ (30)

(ii) Vlassov kinematics

The displacement $u^0$ at the leading order satisfies $u^0 \in G$ or equivalently:

\[
\begin{align*}
\gamma_{22} (u^0) &= 0 \iff \partial_2 u^0_2 - cu^0_3 = 0 \\
\rho_{22} (u^0) &= 0 \iff \partial_2 u^0_2 + \partial_2 u^0_3 + \partial_3 (cu^0_2) - c^2 u^0_3 = 0
\end{align*}
\]

We will now prove that the first two conditions $\gamma_{22} = 0$ and $\rho_{22} = 0$ characterize a rigid solid displacement in the plane $(e_2, e_3)$ of the cross-sections. Indeed, combining the two first relations of (31), we get:

### $\partial_2 [\partial_2 u^0_2 + cu^0_3] = 0$ (32)

which leads to

### $\partial_2 u^0_2 + cu^0_3 = \bar{\sigma}^0$ (33)

where $\bar{\sigma}^0$ only depends on $y_1$ and will represents the twist angle. We then have to integrate the system:

\[
\begin{align*}
\partial_2 u^0_2 - cu^0_3 &= 0 \\
\partial_2 u^0_2 + cu^0_3 &= \bar{\sigma}^0
\end{align*}
\]

To this end, we proceed as in Hamdouni and Millet, in press. Let us first define the angle $\alpha$ between the direction $e_2$ and the tangent vector $a_2$ to the profile (see Fig. 2). It only depends on the curvilinear abscissa $y_2$. Moreover, we have the geometric relations:

### $\frac{dx_2}{dy_2} = \cos \alpha$, $\frac{dx_3}{dy_2} = \sin \alpha$, $\frac{dx_2}{dy_2} = c(y_2)$ (35)

where we recall that $c(y_2)$ denotes the curvature of the profile in the plane of the cross-sections. Now, let us express the components $(u^0_2, u^0_3)$ of the displacement $u^0$ in the local basis $(a_2, a_3)$, with $a_3 = N$, with respect to its components $(U^0_2, U^0_3)$ in the cartesian basis $(e_2, e_3)$. We have:

### $u^0_2 = U^0_2 \cos \alpha + U^0_3 \sin \alpha$ (36)

### $u^0_3 = -U^0_2 \sin \alpha + U^0_3 \cos \alpha$

and (34) then leads to:

### $\frac{\partial U^0_2}{\partial y_2} = -\bar{\sigma}^0 \sin \alpha$, $\frac{\partial U^0_3}{\partial y_2} = \bar{\sigma}^0 \cos \alpha$

By integration with respect to $y_2$, according to (35), we obtain:

\[
\begin{align*}
U^0_2 &= U^0_2 - (x_2 - x'_2) \bar{\sigma}^0 \\
U^0_3 &= U^0_3 + (x_3 - x'_3) \bar{\sigma}^0
\end{align*}
\]

where $U^0_2$ and $U^0_3$ denote the components of the bending displacement, expressed in the cartesian basis $(0, e_2, e_3)$, of an arbitrary point C whose coordinates are $(x'_2, x'_3)$. Finally, replacing expressions (38) of $u^0_2$ and $u^0_3$ in (36), we obtain the kinematics of Result 4.1 for the bending displacements:

\[
\begin{align*}
u^0_2 &= U^0_2 \cos \alpha + U^0_3 \sin \alpha - q(y^0_2) \bar{\sigma}^0 \\
u^0_3 &= -U^0_2 \sin \alpha + U^0_3 \cos \alpha + l(y^0_2) \bar{\sigma}^0
\end{align*}
\]

with:

\[
\begin{align*}
l(y^0_2) &= (x_2 - x'_2) \cos \alpha + (x_3 - x'_3) \sin \alpha \\
q(y^0_2) &= -(x_3 - x'_3) \sin \alpha + (x_3 - x'_3) \cos \alpha
\end{align*}
\]

The kinematics (39) and (40) correspond to the classical Vlassov one\textsuperscript{11} Grillet, 2003; Grillet et al., 2000; Vlassov, 1962; Hamdouni and Millet, in press, where $\bar{\sigma}^0$ is the rotation of the profile around the axis $(C, e_1)$. The functions $l(y^0_2)$ and $q(y^0_2)$ correspond to the coordinates of the vector $Cp$ in the Frenet basis of the profile (Fig. 2).

\textsuperscript{11} Vlassov kinematics is nothing else than a rigid solid kinematics in the plane $(e_2, e_3)$ of the cross-sections written in the Frenet basis $(a_2, a_3)$ of the profile with $a_3 = N$. 

---

\[\text{Fig. 2. Geometric interpretation of } l(y^0_2) \text{ and } q(y^0_2).\]
To finish, the traction or longitudinal displacement $u_1^0$ can be deduced from (39) using the non-distorsion condition $\gamma_{12}(u^0) = 0$.

Let us notice that Vlassov’s kinematics of Result 4.1 is obtained naturally here as a penalty condition, whereas it constitutes generally an assumption imposed a priori Vlassov, 1962. In Volovoi and Hodges, 2000, the kinematics obtained using a simplified expression of the tensor of curvature variation, and a different asymptotic approach is slightly different.

(iii) Associated boundary conditions

We shall now exhibit the boundary conditions associated with Vlassov kinematics. Indeed, the fixing boundary conditions at the two extremities of the beam can be written at the first order:

\[ u_1^0 = u_2^0 = u_3^0 = 0 \quad \text{for} \quad y_1 = 0 \quad \text{and} \quad y_1 = 1 \quad \forall y_2 \in [y_2^1, y_2^2] \]  

where $y_2^1$ and $y_2^2$ are the (curvilinear) coordinates of the extremities of the profile in the direction $y_1$, an origin $y_1^2$ being chosen arbitrary. Moreover, according to the form of the Vlassov kinematics (22) and (23) obtained for $u^0$, the boundary conditions (41) imply that:

\[
\begin{cases}
\begin{align*}
\frac{\partial U_1(0)}{\partial y_1} &= \frac{\partial U_2(0)}{\partial y_1} = 0 \\
\frac{\partial U_3(0)}{\partial y_1} &= \frac{\partial U_4(0)}{\partial y_1} = 0 \\
\frac{\partial U_1(L)}{\partial y_1} &= \frac{\partial U_2(L)}{\partial y_1} = 0 \\
\frac{\partial U_3(L)}{\partial y_1} &= \frac{\partial U_4(L)}{\partial y_1} = 0
\end{align*}
\end{cases}
\]  

(42)

and

\[ U_1(0) = U_2(L) = 0 \]  

(43)

Thus, we have four boundary conditions for the bending displacements $U_1$ and $U_2$ and for the twist angle $\Omega$, but only two for the axial displacement $U_3$. We shall see that this is consistent with the one-dimensional equilibrium equations obtained which are fourth-order differential equations except for the traction equation which is only a second-order differential equation.

Equivalently, using the functional framework associated with the weak formulation of the Koirer model, as $u^0 \in V = \{ u \in H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega) ; \nu \text{ satisfying (41)} \}$, we have:

\[
\begin{align*}
U_1 &\in H^0_0([0, L]), \\
U_2 &\in H^0_0([0, L]), \\
U_3 &\in H^0_0([0, L]), \\
\Omega^0 &\in H^0_0([0, L])
\end{align*}
\]

which will be used later.

(iv) Complementary relation

To finish the proof of Result 4.1, we shall establish a complementary relation which will be useful to establish the one-dimensional equilibrium equations. Let us come back to problem $\varphi^0$. According to (26) and (29), it reduces to:

\[
\int_{\Omega} \left[ |A| \gamma_{12} (u^2) + B \gamma_{11} (u^2) \right] \gamma_{12} (v) \ dS = 0 \forall v \in V
\]  

(44)

Let us now consider the following particular test displacement:

\[ v = (0, v_2, 0) \]  

(45)

with $v_2 \in C^0(\Omega) \subset H^1(\Omega)$, satisfying the boundary conditions (41) ($v_2(0) = v_2(1) = 0$). We then have $\gamma_{12} = \gamma_{22}$ and (44) becomes:

\[
\int_{\Omega} \left[ |A| \gamma_{12} (u^2) + B \gamma_{11} (u^2) \right] v_2 \ dS = 0 \quad \forall v_2 \in C^0(\Omega)
\]  

(46)

In the case of an open cross-section, when $v_2$ is arbitrary, so is $\nu_{22}$. Indeed, we can always take for $v_2$ the primitive of an arbitrary function $w \in C^0(\Omega)$ with respect to $y_2$ which leads to $\nu_{22} = w \in C^0(\Omega)$. This way, relation (46) leads to:

\[ A \gamma_{12} (u^2) + B \gamma_{11} (u^2) = 0 \quad \text{in} \ \Omega \]  

(47)

This ends the proof of Result 4.1.

5. Weak formulation in the subspace $G$

According to (26) and (29) obtained previously as penalty conditions, problem $\varphi^0$ writes:

\[
\int_{\Omega} \left[ |B| \gamma_{12} (u^2) + A \gamma_{11} (u^2) \right] \gamma_{11} (v) + B \gamma_{12} (u^2) \gamma_{12} (v) + A \gamma_{12} (u^2) \gamma_{22} (v) \ dS + \frac{1}{3} \int_{\Omega} \left( 3 \rho_{11} (u^2) \rho_{11} (v) + 3 \rho_{12} (u^2) \rho_{12} (v) \right) \ dS = b(v) \quad \forall v \in V
\]  

(48)

with

\[ b(v) = \int_{\Gamma} f_1 v_1 + f_2 v_2 + f_3 v_3 \ dS \]

Taking test functions $v \in G$ (i.e submitted to the same constraints as $u^0$), and using relation (47), problem $\varphi^0$ reduces to:

\[
\int_{\Omega} 2 E \gamma_{11} (u^0) \gamma_{11} (v) \ dS + \int_{\Omega} 8 \rho_{12} (u^0) \rho_{12} (v) \ dS = \int_{\Gamma} f v \ dS \quad \forall v \in G
\]  

(49)

Moreover, according to Result 4.1, the displacement $u^0$ satisfies Vlassov kinematics (22) and (23) and we have:

\[
\gamma_{11} (u^0) = \frac{dU_1}{dy_1} - x_2 \frac{d^2 U_3}{dy_2^2} - x_3 \frac{d^2 U_4}{dy_3^2} - \omega \frac{d\Omega}{dy_1}
\]

(50)

On the other hand, from (22), using the geometric relations (35), expression of $\rho_{12}(u^0)$ given from (19) reduces to:

\[
\rho_{12}(u^0) = \frac{d}{dy_2} - c q
\]

(51)

Finally, from definition (23) of $l(y_2)$ and $q(y_2)$, using again (35), we have:

\[
\frac{d}{dy_2} - c q = 1
\]

and finally:

\[
\rho_{12}(u^0) = \frac{d\Omega}{dy_1}
\]

(52)

Obviously, as $v \in G$, it writes:

\[
\nu = \left( \begin{array}{c}
- \nabla_1 x_2 \frac{\partial \nu_4}{\partial y_1} - x_2 \frac{\partial \nu_4}{\partial y_1} - \omega \frac{\partial \nu_3}{\partial y_1} \\
\nabla_2 \cos x + \nabla_3 \sin x - q(y_2) \delta \\
- \nabla_3 \sin x + \nabla_3 \cos x + l(y_2) \delta
\end{array} \right)
\]

(53)

where $(\nabla_1, \nabla_2, \nabla_3, \delta)$ denotes an arbitrary field of test displacements depending only on $y_1$. Obviously, we have:

\[
\begin{align*}
\gamma_{11} (v) &= \frac{\partial \nu_1}{\partial y_1} - x_2 \frac{\partial \nu_4}{\partial y_1} - x_3 \frac{\partial \nu_4}{\partial y_1} - \omega \frac{\partial \nu_3}{\partial y_1} \\
\rho_{12} (v) &= \frac{\partial \Omega}{\partial y_1}
\end{align*}
\]

(54)

which will be used in the sequel to establish the one-dimensional equilibrium equations.

6. One-dimensional equilibrium equations

6.1. Traction equation

**Result 6.1.** For applied force levels such as $G_2 = G_3 = \omega^0$ and $G_1 = \omega^5$, the unknowns $(U_1, U_2, U_3, \Omega^0)$ are solutions of the following traction equation:
The weak formulation (55) becomes:

\[
\frac{2E}{\mu} \int \left( \frac{d\mathbf{U}_1}{dy_1} - x_2 \frac{d^2\mathbf{U}_2}{dy_1^2} - x_3 \frac{d^2\mathbf{U}_3}{dy_1^2} - \alpha \frac{d^2\bar{\mathbf{S}}}{dy_1^2} \right) \left( -\alpha \frac{d^2\bar{\mathbf{S}}}{dy_1^2} \right) dy_1 dy_2 \\
+ \frac{8}{3} \int \left( -\alpha f_1 \frac{d\delta}{dy_1} - q(y_2) f_3 \delta + l(y_2) f_1 \delta \right) dy_1 dy_2 \\
\forall \delta \in H^2_0([0,1])
\]  

(59)

After two successive integrations by parts with respect to \( y_1 \), for the first term of the left hand side, and only one for the second term, as \( \delta \) is arbitrary in \( H^2_0([0,1]) \), we obtain the twist Eq. (58).

6.3. Bending equation in the direction \( e_2 \)

**Result 6.3.** For applied force levels such as \( G_2 = G_3 = \rho^e_2 \) and \( G_1 = \rho^e_3 \), the unknowns \( \{\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3, \bar{\mathbf{S}}\} \) are solutions of the following bending equation in the direction \( e_2 \):

\[
- \frac{E}{\mu} \frac{d^2\mathbf{U}_1}{dy_1^2} + \frac{E}{\mu} \frac{d^2\mathbf{U}_2}{dy_1^2} + \frac{E}{\mu} \frac{d^2\mathbf{U}_3}{dy_1^2} + \frac{E}{\mu} \frac{d^2\bar{\mathbf{S}}}{dy_1^2} = F_2 + \frac{dM_{12}}{dy_1^2}
\]  

(60)

with

\[
S_2 = 2 \int_{y_2}^{y_1} x_2 dy_2 \quad J_{22} = 2 \int_{y_2}^{y_1} x_2^2 dy_2 \quad J_{23} = 2 \int_{y_2}^{y_1} x_2 x_3 dy_2
\]

\[
J_{20} = 2 \int_{y_2}^{y_1} \alpha x_2 dy_2 \quad F_2 = \int_{y_2}^{y_1} [f_2 \cos \alpha - f_3 \sin \alpha] dy_2
\]

\[
M_{12} = \int_{y_2}^{y_1} x_2 f_1 dy_2
\]

**Proof.** To establish this bending equation, we shall take in (49) test displacements verifying (53) with \( \nabla_1 = \nabla_3 = \delta = 0 \), and where \( \nabla_2 \in H^2_0([0,1]) \) is arbitrary. This implies that \( \nu = \left( -\frac{d\nu_1}{dy_1} x_2, \nabla_2 \cos \alpha, \nabla_2 \sin \alpha \right) \), so that

\[
\gamma_{11}(\nu) = -\frac{d^2\nu_2}{dy_1^2} \quad \text{and} \quad \rho_{12}(\nu) = 0
\]  

(61)

The weak formulation (49) then writes:

\[
2 \int \left( \frac{d\mathbf{U}_1}{dy_1} - x_2 \frac{d^2\mathbf{U}_2}{dy_1^2} - x_3 \frac{d^2\mathbf{U}_3}{dy_1^2} - \alpha \frac{d^2\bar{\mathbf{S}}}{dy_1^2} \right) \left( -\alpha \frac{d^2\bar{\mathbf{S}}}{dy_1^2} \right) dy_1 dy_2
\]

\[
= \int \left[ -f_1 x_2 \frac{d\nu_2}{dy_1^2} + f_1 \nabla_2 \cos \alpha - f_3 \nabla_2 \sin \alpha \right] dy_1 dy_2 \\
\forall \nabla_2 \in H^2_0([0,1])
\]

As previously, after two successive integrations by parts with respect to \( y_1 \), we obtain the bending equation (60) in the direction \( e_2 \).

6.4. Bending equation in the direction \( e_3 \)

**Result 6.4.** For applied force levels such as \( G_2 = G_3 = \rho^e_2 \) and \( G_1 = \rho^e_3 \), the unknowns \( \{\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3, \bar{\mathbf{S}}\} \) are solutions of the following bending equation in the direction \( e_3 \):

\[
- \frac{E}{\mu} \frac{d^2\mathbf{U}_1}{dy_1^2} + \frac{E}{\mu} \frac{d^2\mathbf{U}_2}{dy_1^2} + \frac{E}{\mu} \frac{d^2\mathbf{U}_3}{dy_1^2} + \frac{E}{\mu} \frac{d^2\bar{\mathbf{S}}}{dy_1^2} = F_3 + \frac{dM_{13}}{dy_1^2}
\]  

(62)

with

\[
S_3 = 2 \int_{y_2}^{y_1} x_3 dy_2 \quad J_{33} = 2 \int_{y_2}^{y_1} x_2 x_3 dy_2 \quad J_{23} = 3 \int_{y_2}^{y_1} x_3^2 dy_2 \\
J_{20} = 2 \int_{y_2}^{y_1} \cos \alpha x_2 dy_2 \quad F_3 = \int_{y_2}^{y_1} [f_2 \sin \alpha + f_3 \cos \alpha] dy_2
\]

\[
M_{13} = \int_{y_2}^{y_1} x_2 f_1 dy_2
\]

**Proof.** Let us now consider particular test displacements (53) such that \( \nabla_1 = \nabla_3 = \delta = 0 \) and \( \delta \) is arbitrary in \( H^2_0([0,1]) \). Then we have

\[
\nu = \left( -\alpha \frac{d\nu_2}{dy_1}, -q(y_2) \delta, l(y_2) \delta \right) \quad \text{so that}
\]

\[
\gamma_{11}(\nu) = -\alpha \frac{d^2\nu_2}{dy_1^2} \quad \text{and} \quad \rho_{12}(\nu) = \frac{d\delta}{dy_1}
\]

The weak formulation (49) becomes:
**Proof.** Finally, let us consider the test displacements (49) with \( \mathbf{v}_1 = \mathbf{v}_2 = \hat{\mathbf{v}} = 0 \) and \( \mathbf{v}_3 \in H^2_0(\{0, 1\}) \) arbitrary. We then have
\[
v = \left(-\frac{\partial^2 \mathbf{v}_3}{\partial y^2}ight) x_1 \quad \text{and} \quad \rho_{12}(v) = 0
\]
Thus, the weak formulation (49) reduces to:
\[
\begin{align*}
2E \frac{\mu}{\mu} \int_0^1 & \left( \frac{d\mathbf{U}_1}{dy_1} - x_2 \frac{d^2\mathbf{U}_2}{dy_1^2} - x_3 \frac{d^2\mathbf{U}_3}{dy_1^2} - \frac{\partial \mathbf{\sigma}_p}{\partial y_1} \right) \left(-x_1 \frac{d^2\mathbf{v}_3}{dy_1^2} dy_1 \right. \\
& \left. \quad \nabla \mathbf{v}_3 \mathbf{e}^H_3(\{0, 1\}) \right) dy_1 dy_2 \\
& \quad = \int_0^1 \left[ -f_1 x_3 \frac{d^2\mathbf{v}_3}{dy_1^2} + f_2 \mathbf{v}_3 \sin \varphi + f_3 \mathbf{v}_3 \cos \varphi \right] dy_1 dy_2 \\
& \quad = \int_0^1 \frac{\partial \mathbf{\sigma}_p}{\partial dy_1} dy_1 dy_2
\end{align*}
\]
and leads, after two successive integrations by parts with respect to \( y_1 \), as \( \mathbf{v}_3 \) is arbitrary in \( H^2_0(\{0, 1\}) \), to the bending equation (62) in the direction \( e_1 \).

**7. Reduced dimensional equilibrium equations and comparison with Vlassov model**

### 7.1. Reduced dimensional equilibrium equations

The four one-dimensional equilibrium equations obtained at Results 6.1–6.3 and 6.4 were obtained in any cartesian frame \( \mathcal{F} = \{O, e_1, e_2, e_3\} \) and involve the displacement of a point \( C \) arbitrary chosen in the plane of the section. However, they can be simplified considering a particular choice of the frame \( \mathcal{F} \) and of \( C \). Indeed, if \( \mathcal{F} \) corresponds to the main inertia frame of the profile, we classically have:
\[
S_2 = S_3 = 0 \quad \text{and} \quad J_{23} = 0
\]
Moreover, choosing the point \( C \) as the shear center of the section, which will be used as the pole for the computation of the sectorial area \( \hat{\omega}_s \) (see Vlassov, 1962, chapter 1), we have:
\[
J_{23} = J_{32} = 0
\]
Finally, the cancellation of the sectorial static moment \( S_\omega \), determines the position of the origin of the curvilinear abscissa \( y_2 \). If the profile has an axis of symmetry, the latter must be chosen at the intersection between the axis of symmetry and the profile. With such a parametrization, the dimensionless equilibrium equations of Results 6.1–6.3 and 6.4 reduce to:
\[
-\frac{E_S}{\mu} \frac{d^2\mathbf{U}_1}{dy_1^2} = F_1
\]
\[
-\frac{E_J_{\omega 0}}{\mu} \frac{d^2\mathbf{U}_2}{dy_1^2} + J_{\omega d} \frac{d^2\mathbf{\sigma}_p}{dy_1^2} = M_t + dM_1
\]
\[
-\frac{E_J_{12}}{\mu} \frac{d^2\mathbf{U}_3}{dy_1^2} = F_2 + dM_{12}
\]
\[
-\frac{E_J_{13}}{\mu} \frac{d^2\mathbf{U}_3}{dy_1^2} = F_3 + dM_{13}
\]
We are now going back to dimensional variables in the reduced equilibrium equations above, for a comparison with Vlassov model. To this end, we define:
\[
\mathbf{U}_1 = \mathcal{E}_h \mathbf{U}_1, \quad \mathbf{U}_2 = \mathcal{E}_h \mathbf{U}_2, \quad \mathbf{U}_3 = \mathcal{E}_h \mathbf{U}_3, \quad \mathbf{\sigma}_p = \mathcal{E}_h \mathbf{\sigma}_p
\]
\[
\hat{x}_1 = \lambda_1 \hat{x}_1, \quad \hat{x}_2 = \hat{x}_2, \quad \hat{x}_3 = \hat{x}_3, \quad \hat{\omega} = \hat{\omega} \quad \hat{q} = \hat{q} \quad \dot{\lambda} = d \lambda
\]
We then have the following result:

**Result 7.1.** For applied force levels such as \( G_2 = G_3 = \hat{v}^\theta \) and \( G_1 = \hat{v}^\theta \), the unknowns \( \mathbf{U}_1, \mathbf{\sigma}_p, \mathbf{U}_2, \mathbf{U}_3 \) are solutions of the following reduced equilibrium equations:

\[
-\frac{E_S}{\mu} \frac{d^2\mathbf{\mathbf{U}}_1}{dy_1^2} = \mathbf{F}_1
\]
\[
E_J_{\omega 0} \frac{d^2\mathbf{\sigma}_p}{dy_1^2} - J_{\omega d} \frac{d^2\mathbf{\sigma}_p}{dy_1^2} = M_t + dM_1
\]
\[
-\frac{E_J_{12}}{\mu} \frac{d^2\mathbf{U}_3}{dy_1^2} = F_2 + dM_{12}
\]
\[
-\frac{E_J_{13}}{\mu} \frac{d^2\mathbf{U}_3}{dy_1^2} = F_3 + dM_{13}
\]
where the dimensional geometric constants and forces are given by:
\[
\mathbf{\bar{S}} = 2h \int_{y_2}^{\hat{y}_2} d\bar{y}_2 \quad \mathbf{\bar{J}}_{\omega 0} = 2h \int_{y_2}^{\hat{y}_2} \hat{\omega} d\bar{y}_2 \quad \mathbf{\bar{J}}_{\omega d} = \frac{8}{3} h \int_{y_2}^{\hat{y}_2} d\bar{y}_2
\]
\[
\mathbf{\bar{F}}_1 = \int_{y_2}^{\hat{y}_2} \hat{f}_1 d\bar{y}_2 \quad \mathbf{\bar{F}}_2 = \int_{y_2}^{\hat{y}_2} \left[ f_2 \cos \varphi - f_3 \sin \varphi \right] d\bar{y}_2
\]
\[
\mathbf{\bar{F}}_3 = \int_{y_2}^{\hat{y}_2} \left[ f_2 \sin \varphi + f_3 \cos \varphi \right] d\bar{y}_2 \quad M_t = \int_{y_2}^{\hat{y}_2} \left[ \hat{y}^2 \hat{f}_3 - \hat{q} \hat{y}_2 \hat{f}_2 \right] d\bar{y}_2
\]
\[
M_1 = \int_{y_2}^{\hat{y}_2} \hat{q} \hat{f}_1 d\bar{y}_2 \quad M_{12} = \int_{y_2}^{\hat{y}_2} \hat{x}_2 \hat{f}_1 d\bar{y}_2 \quad M_{13} = \int_{y_2}^{\hat{y}_2} \hat{x}_3 \hat{f}_1 d\bar{y}_2
\]
The proof does not constitute any difficulty and is left to the reader.

### 7.2. Comparison with Vlassov model and discussion

Vlassov equilibrium equations for thin-walled beams and the associated kinematics were established in Vlassov, 1962 using a priori kinematics and statics assumptions, by considering the equilibrium of an infinitesimal piece of beam of length \( dx \). In this paper, the kinematics Result 4.1 and the dimensional equilibrium equations (70)–(73) are deduced rigorously from the asymptotic expansion of Koiter shell model for very long cylindrical shells. The kinematics and the associated one-dimensional equations obtained corresponds to the Vlassov ones, except for the geometric constant \( J_{\omega d} \) present in the second-order term of the twist equation (71). Indeed, Result 7.1 gives the following general analytical expression:
\[
\mathbf{\bar{J}}_{\omega d} = \frac{8}{3} h \int_{y_2}^{\hat{y}_2} d\bar{y}_2 = \frac{8}{3} h \bar{d}
\]
where \( d \) is the length of the profile. In Vlassov, 1962, Vlassov only gives an empirical expression for this geometric constant noted \( J_{\omega d} \):
\[
\mathbf{\bar{J}}_{\omega d} = \frac{8a}{3} \frac{h}{\bar{d}}
\]
where \( a \) is an empirical constant which must be determined experimentally. For a half-cylinder of radius \( R \) and thickness \( 2h \), the experiments carried out by Vlassov give a mean value of \( a \) equal to 0.99 (see Vlassov, 1962, p. 134). In this particular case, expressions (74) and (75) nearly confuse. However, with the empirical expression (75), for each profile it is necessary to carry out new
experiments in order to determine the specific mean value of \( a \). Conversely, with the asymptotic approach developed here, we obtain an exact analytical expression (74) of \( J_{e,f} \), whose computation is easy for any profile.

The one-dimensional equilibrium equations for thin-walled beams which are obtained from the asymptotic expansion of Koiter model in this paper, do not correspond exactly to those obtained in Grillet, 2003; Hamdouni and Millet, in press from a direct asymptotic expansion of the three-dimensional linear elasticity equations, although the calculations have been performed for the same geometry and the same level of applied forces (described accurately by the dimensional numbers introduced). Even if the one-dimensional traction and twist equations obtained are identical, the bending equations slightly differ. In Grillet, 2003; Hamdouni and Millet, in press, we obtained a supplementary term coupling bending and twist effects in the bending equations. Note that this coupling is different from that obtained in Volovoi and Hodges, 2000 which is due to the anisotropic properties of the material and does not exist for isotropic elastic shells. This result can be explained by the fact that the three-dimensional equations are “richer” than the two-dimensional Koiter shell model (they contain in particular transversal shear whereas Koiter model does not), and lead to a supplementary term coupling bending and twist.

However, as the Koiter shell model is not itself an asymptotic model, even if it degenerates to the Vlassov thin-walled beam model for very long shells, we cannot conclude that the Vlassov model is an asymptotic model.

8. Conclusion

In this paper, we performed an asymptotic expansion of the Koiter shell model when the ratio width/length of the middle surface tends to zero. We proved that the Koiter model degenerates, for very long cylindrical shells with open cross-section (particular case of tubes were not considered), to a one-dimensional thin-walled beam model whose kinematics and equilibrium equations correspond to the Vlassov ones. The interest of the asymptotic analysis performed in this paper from the Koiter shell model is double. First, the result obtained proves that a very long cylindrical shell described by Koiter model behaves like a thin-walled beam with open cross-section described by Vlassov model. In other terms, we can use indifferently Koiter shell model or Vlassov model to describe such long and thin structures, with strongly bent open cross-section. On the other hand, with the rigorous derivation of the one-dimensional thin-walled beam equations from Koiter model, we obtained a general analytical expression of the geometric constant \( J_{e,f} \) involved in the twist equation, which can be calculated very easily for any profile. This analytical expression of the twist rigidity \( J_{e,f} \) improves the expression proposed by Vlassov, which depends on an empirical constant whose experimental determination must be performed for each profile considered.

References


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