Compatibility of large deformations in nonlinear shell theory

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Abstract - In this paper, we propose to establish compatibility conditions satisfied by the membrane strain tensor $\Delta$ and the curvature strain tensor $K$ of a shell midsurface. To this end, we use the polar decomposition $RU$ of the linear tangent map of a mapping function $\phi$. First, we characterize the tangential part of the derivative of $R$ by a new variable $\Lambda$. Then the compatibility equations are decomposed in a natural way into a first-order differential system. The use of $\Lambda$ greatly simplifies the differential calculus and gives an intrinsic and explicit formulation for compatibility conditions in nonlinear shell theory.

compatibility equations / shell theory / large deformations

1. Introduction

The equilibrium equations of nonlinear shell theory are frequently set in terms of displacements. This kind of formulation is inevitable when boundary conditions include imposed displacements. Nevertheless, in some particular cases (e.g. spherical deformation of a spherical shell, cylindrical deformation of a cylindrical shell etc.), the shell equations may advantageously be set in terms of strain measures. In this case, it is necessary to add compatibility conditions to the shell equilibrium equations.

As is well known, the compatibility equations of a three-dimensional continuum media are of second order (e.g. Truesdell and Toupin, 1960). Recently, Edelen (1990) and Vallée (1992) have given a first-order version of these equations. In Vallée (1992) for example, the author uses the polar decomposition $RU$ to separate compatibility conditions into a first-order differential system. In addition, he gives a construction method of the deformed shell midsurface.

Contrary to the three-dimensional case, there exist only a few works dealing with compatibility conditions in shell theory. In Valid (1973), the compatibility conditions for linearized strain tensors were presented. In Breuneval (1972), these conditions were obtained in the nonlinear case. Nevertheless, the formulation proposed makes use of an operator of connection and another of curvature which are not explicitly expressed in terms of shell strain measures.

More recently, Vallée and Fortuné (1996) have used polar decomposition to reformulate the Gauss-Codazzi equations of a surface. The formulation so obtained is explicit and very elegant.

The purpose of the present investigation is to generalize Vallée’s work (Vallée and Fortuné, 1996) to compatibility conditions in the nonlinear Kirchhoff–Love shell theory. To this end, we use the polar decomposition $RU$ of the linear tangent map $R_{\partial \phi / \partial \phi_0}$ of a mapping function $\phi$. Studying variations of $R$, we introduce a new

* Correspondence and reprints
variable $\Lambda$ which simplifies intrinsic differential calculus along the reference shell midsurface. This method leads in a natural way to a first-order differential system. Moreover, we give an explicit expression for the Riemannian curvature operator of the mapping function $\phi$ in terms of $\Lambda$ and $U$.

To end the article, we show how to construct the deformed shell when the compatibility conditions are satisfied. The described method is divided into clearly separated steps.

2. Notations and preliminaries

In this section most notations, definitions and results may be found in Breuneval (1972) and Valid (1995).

Let $V$ be a smooth two-dimensional Riemannian manifold embedded in the $\mathbb{R}^3$ space. For $X$ and $dp \in T_pV$ (the tangent plane), we note $\hat{d}X$ to be the covariant derivative of $X$ along $dp$. This derivative can be defined as $\hat{d}X = \Pi dX$ where $\Pi$ is the orthogonal projector from $\mathbb{R}^3$ onto the tangent plane $T_pV$. We also define the covariant derivative of a one-form field and a tensor field (Breuneval, 1972; Valid, 1995).

In the present paper, we will use the following notations.

- $\sigma$ denotes the two-form field along $V$, defined as:

$$\sigma(dp, dp) = N(dp \wedge dp) = dp N dp \quad \forall \, dp, \, dp \in T_pV$$  \hspace{1cm} (1)

where $N$ is the unit normal to $V$, the overbar the transposition operator and $\Sigma$ the $\frac{\pi}{2}$-rotation around the normal.

- For all vector fields $\Lambda$ along $V$, we define its scalar curl as:

$$\text{curl}(\Lambda) \sigma(dp, dp) = \overline{(d\Lambda dp - \delta \Lambda dp)} \quad \forall \, dp, \, dp \in T_pV$$  \hspace{1cm} (2)

- For all tensor fields $A$ along $V$, we define its curl which is a one-form field along $V$, as

$$\text{Curl}(A) \sigma(dp, dp) = \overline{dp dA - dp \delta A} \quad \forall \, dp, \, dp \in T_pV$$  \hspace{1cm} (3)

- If $V_0$ is another smooth manifold, we will index the geometrical objects associated with $V_0$ by zero.

- Let $\phi$ be a smooth map from $V_0$ onto $V$, we note $\frac{\partial \phi}{\partial p_0}$ the linear tangent map to $\phi$ (well known as the deformation gradient of the shell).

- The derivative of a linear mapping field $A$ defined from $T_{p_0}V_0$ onto $T_pV$ ($p = \phi(p_0)$) can be put in the following form

$$dA = \hat{d}A - N \overline{\delta A N_0} - AdN_0N_0 = \hat{d}A + N \overline{d_p C A} + AC_0 dp_0 N_0$$  \hspace{1cm} (4)

where $dp = \frac{\partial \phi}{\partial p_0} dp_0$ and $C = -\frac{\partial N}{\partial p}$ (respectively $C_0 = -\frac{\partial N_0}{\partial p_0}$) denotes the curvature operator of $V$ (respectively $V_0$).

Remark 1: in the previous formulae, $\hat{d}A$ is defined as $\hat{d}A = \Pi dA N_0$. This notation will also be used when $A$ is a tensor field along $V$ or $V_0$. For example, if $A$ is a tensor field along $V$, $\hat{d}A = \Pi dA N_0$. In particular, we have classically: $\hat{d}N = \hat{d} \Sigma = \hat{d} I_{TV} = 0$ ($I_{TV}$ is the identity operator of the tangent plane $T_p V$).

- Finally, for all endomorphisms $A$ of $T_pV$, we will use the following formulae

$$\text{Adj}(A) = \Sigma A \Sigma, \quad \det(A) \Sigma = A \Sigma A = \overline{A \Sigma A}$$

$$\text{Adj}(A) A = A \text{Adj}(A) = \det(A) I_{TV}$$  \hspace{1cm} (5)

3. Compatibility conditions
Let \( V_0 \) be a smooth two-dimensional Riemannian manifold embedded in the \( \mathbb{R}^3 \) space. We suppose \( V_0 \) to be oriented and simply connected. \( V_0 \) is taken as the reference configuration of the shell midsurface. The shell deformation is then described by a smooth functional mapping \( \phi \) defined from \( V_0 \) onto another manifold \( V \) (the deformed shell midsurface).

Within the framework of the nonlinear shell theory, which does not include the transverse shear strain, the shell deformation is classically measured thanks to the membrane strain tensor \( \Delta \) and to the curvature strain tensor \( K \) defined as

\[
\Delta = \frac{\partial \mu}{\partial p_0} \frac{\partial \mu}{\partial p_0} \quad \text{and} \quad K = \frac{\partial \mu}{\partial p_0} C \frac{\partial \mu}{\partial p_0}
\]

The following development is based on the polar decomposition of the deformation gradient. As is well known, the linear mapping field \( \frac{\partial \mu}{\partial p_0} \) possesses a unique polar decomposition of the form

\[
\frac{\partial \mu}{\partial p_0} = RU
\]

where \( U \) is the positive definite square root of \( \Delta \) and \( R = \frac{\partial \mu}{\partial p_0} U^{-1} \) is a linear mapping field defined from \( T_{p_0} V_0 \) onto \( T_p V \). So we have

\[
U^2 = \frac{\partial p}{\partial p_0} \frac{\partial p}{\partial p_0}, \quad \vec{R} = I_{T_{V_0}} \quad \text{and} \quad R\vec{R} = I_{TV}
\]

The compatibility problem we propose to solve is as follows: given two symmetrical tensor fields \( U \) and \( K \) along \( V_0 \) such that \( U \) is positive definite, which conditions must they satisfy to make integrable the following equation

\[
dp = RU dp_0
\]

where \( p = \phi(p_0) \) and \( \phi \) is a smooth functional mapping defined from \( V_0 \) onto \( V \).

To this end, we will first study the variations of \( R \) and then the variations of \( p \).

**Remark 2:** since \( V_0 \) is a smooth Riemannian manifold embedded in the \( \mathbb{R}^3 \) space, we have \( d\delta p_0 = \delta dp_0 \) and \( d\delta N_0 = \delta dN_0 \) (these relations are equivalent to the Gauss–Codazzi equations).

### 3.1. Variations of \( R \)

The first point that should be made is to characterize the derivative of \( R \). The following lemma gives the form of the tangential part of this derivative.

**Lemma 1:** there exists a one-form field \( \vec{\alpha} \) along \( V_0 \) such that

\[
\hat{d}R = (\vec{\alpha} dp_0) \hat{\Sigma}_0
\]

**Proof:** since \( \vec{R} \vec{R} = I_{T_{V_0}} \), we have

\[
\overrightarrow{dR} R + \overrightarrow{R} dR = 0 \quad (\hat{d} I_{T_{V_0}} = 0)
\]

The linear tangent mapping \( \overrightarrow{dR} R \) is therefore an antisymmetrical endomorphism of the tangent plane \( T_{p_0} V_0 \). As the vectorial space of these endomorphisms is of one dimension, there exist a scalar field \( \alpha \) such that
\( \dot{R} \dot{d}R = \alpha \Sigma_0 \). Moreover, it is clear that \( \alpha \) depends linearly on \( dp_0 \). Consequently, \( \dot{R} \dot{d}R \) may be expressed as \( \dot{R} = (\ddot{A}dp_0)R \Sigma_0 \) where \( \Lambda \in T_{p_0}V_0 \).

**Interpretation of \( \Lambda \):** the one-form \( \Lambda \) defined as \( (\ddot{A}dp_0) \Sigma_0 = \dot{R} \dot{d}R \) represents the local rotation rate around the normal of a small material element of the shell midsurface.

According to lemma 1 and Eq. (4), the derivative of \( R \) can be put into the following form:

\[
\dot{d}R = (\ddot{A}dp_0)R \Sigma_0 - N d\delta N R - R dN_0 N_0 = (\ddot{A}dp_0)R \Sigma_0 + N \ddot{d}p CR + R C_0 dp_0 N_0
\]

This relation is integrable if and only if the second derivative of \( R \) is symmetrical. The following theorem gives this condition in terms of \( \Lambda, U \) and \( K \).

**Theorem 1:** the relation (6) is integrable if and only if

\[
curl(\Lambda) = det(C_0) - det(K U^{-1}) \quad \text{and} \quad \overline{curl}(K U^{-1}) = Adj(K U^{-1}) \Lambda
\]

**Proof:** on the one hand, the condition \( \ddot{d}R - \delta \dot{d}R = 0 \) may be decomposed into the four following relations

\[
\Pi(\ddot{d}R - \delta \dot{d}R) \Pi_0 = 0
\]

\[
\Pi(\ddot{d}R - \delta \dot{d}R) N_0 N_0 = 0
\]

\[
N \overline{N}(\ddot{d}R - \delta \dot{d}R) \Pi_0 = 0
\]

and

\[
N \overline{N}(\ddot{d}R - \delta \dot{d}R) N_0 N_0 = 0
\]

On the other hand, calculating \( \ddot{d}R - \delta \dot{d}R \), we obtain

\[
\begin{align*}
\ddot{R} \delta R - \delta \dot{d}R &= (\ddot{A}dp_0 - \ddot{d}A\delta p_0)R \Sigma_0 + ((\ddot{A}dp_0)\delta R - (\ddot{A}\delta p_0)dR) \Sigma_0 + R((\ddot{A}dp_0)\delta \Sigma_0 - (\ddot{A}\delta p_0)d \Sigma_0) \\
&- (\delta N d\delta N - dN \delta \delta N) R - N (\delta d\delta N - d\delta \delta N) R - N (\delta d\delta N - d\delta \delta N) R + R(\delta N_0 d\delta N_0 - dN_0 \delta \delta N_0) \\
&- (\delta R dN_0 - dR \delta N_0) \overline{N}_0
\end{align*}
\]

Relation (8) gives

\[
(\ddot{A}dp_0 - \ddot{d}A\delta p_0)R \Sigma_0 + ((\ddot{A}dp_0)\delta R - (\ddot{A}\delta p_0)dR) \Sigma_0 + (\delta N d\delta N - dN \delta \delta N) R + R(\delta N_0 d\delta N_0 - dN_0 \delta \delta N_0) = 0
\]

because \( \dot{d} \Sigma_0 = 0 \).

As \( \ddot{R} = (Adp_0)R \Sigma_0, dN = -Cdp \) and \( dN_0 = -C dp_0 \), we have

\[
(\ddot{A}dp_0 - \ddot{d}A\delta p_0)R \Sigma_0 - C(\delta p_0 dp_0 - dp_0 \delta p_0)CR + R C_0 (dp_0 \delta p_0 - d p_0 \delta p_0) C_0 = 0
\]

Therefore, replacing \( dp_0 \) with \( HU dp_0 \), we get

\[
(\ddot{A}dp_0 - \ddot{d}A\delta p_0)R \Sigma_0 - C \frac{\partial \delta}{\partial p_0} (\delta p_0 dp_0 - dp_0 \delta p_0) \frac{\partial p}{\partial p_0} CR + R C_0 (dp_0 \delta p_0 - d p_0 \delta p_0) C_0 = 0
\]

and using (2) and (5) we obtain

\[
curl(\Lambda) = det(C_0) - det(K U^{-1})
\]
Relation (9) gives
\[ R((\Lambda dp_0)\delta \Sigma_0 - (\Lambda \delta p_0)d\Sigma_0)N_0 - (\delta R N_0 - dR \delta N_0)N_0 = 0 \]
Since \(d\Sigma_0 N_0 = -\Sigma_0 dN_0 N_0\) and \(d\delta R = (\Lambda dp_0)R\Sigma_0\), the previous relation always holds.

Relation (10) gives
\[ ((\Lambda dp_0)N\delta R - (\Lambda \delta p_0)NdR)\Sigma_0 = N(\delta dN - d\delta N)R - N(dN\delta R - \delta N dR) = 0 \]
As \(Nd\delta R = (\Lambda \delta p_0)NdNR\Sigma_0 = -(\Lambda dp_0)N\delta R\Sigma_0\), we get
\[ (\delta dN - d\delta N)R = 0 \]
which is equivalent to
\[ \delta p_0 \delta (\frac{\partial p}{\partial p_0}C)R - \delta p_0 \delta (\frac{\partial C}{\partial p_0})R = 0 \]
Replacing \(dp\) with \(RUdp_0\), we find
\[ \delta p_0 \delta (KU^{-1}) - \delta p_0 \delta (KU^{-1}) = (\delta p_0(\Lambda \delta p_0) - \delta p_0(\Lambda dp_0))KU^{-1}\Sigma_0 \]
Eventually, using (3) and (5) we get
\[ \text{Curl}(KU^{-1}) = \text{Adj}(KU^{-1})\Lambda \]

Relation (11) gives
\[ -N(d\delta N - \delta NdR)N_0 N_0 - NN(dR \delta N_0 - dR \delta N_0)N_0 = 0 \]
Since \(dRN_0 N_0 = -RdN_0 N_0\) and \(NN dR = -NdN R\), the previous relation always holds.

3.2. The variations of \(p\)
Now the complete set of compatibility conditions can be obtained by studying the integrability of \(dp = RUdp_0\). The following theorem gives all the integrability conditions in terms of \(\Lambda\), \(U\) and \(K\).

**Theorem 2:** \(\Lambda = U^2\) and \(K\) represent respectively the membrane and the curvature strain tensor of a shell midsurface if and only if
\[ \text{Curl}(KU^{-1}) = \text{Adj}(KU^{-1})\Lambda \] (12)
and
\[ \text{curl}(\Lambda) = \text{det}(C_0) - \text{det}(KU^{-1}) \] (13)
with
\[ \Lambda = \frac{U}{\text{det}(U)}\text{Curl}(U) \] (14)

**Proof:** the Eqs (12) and (13) are given by Theorem 1. So we have just to prove the relation (14).
On the one hand, the condition $d\delta p - \delta dp = 0$ can be decomposed into

$$\Pi(d\delta p \quad \delta dp) = 0$$  \hspace{1cm} (15)

and

$$NN(d\delta p - \delta dp) = 0$$  \hspace{1cm} (16)

On the other hand, we have

$$d\delta p - \delta dp = d(R\delta p_0) - \delta(RUdp_0)$$

and from $d\delta p_0 - \delta dp_0 = 0$ and $d\delta R - \delta dR = 0$ we get

$$d\delta p - \delta dp = R(\delta U\delta p_0 - \delta Udp_0) + (dR)U\delta p_0 - (\delta R)Udp_0$$

Equation (15) gives

$$R(\delta U\delta p_0 - \delta Udp_0) + (dR)U\delta p_0 - (\delta R)Udp_0 = 0$$

which is equivalent to

$$\delta p_0 \delta U - \delta p_0 \delta U = (\delta p_0(\Lambda\delta p_0) - \delta p_0(\Lambda dp_0))U\Sigma_0$$

Therefore, using (3) and (5) we get

$$\text{Curl}(U) = \text{Adj}(U)\Lambda \quad \text{or} \quad \Lambda = \frac{I}{\det(U)}\text{Curl}(U)$$

Equation (16) gives

$$N(\delta NRU\delta p_0 - \delta NRUdp_0) = 0$$

which yields

$$(d\delta p - \delta dp) = 0 \quad \text{or} \quad (dpC\delta p - \delta pCdp) = 0$$

This relation always holds because $C$ is symmetrical.

4. Example

Let $V_0$ be a circular cylinder defined by the following parametric equation

$$p_0 = (x_1, x_2, x_3), \quad x_1 = r\cos(\theta), \quad x_2 = r\sin(\theta) \quad \text{and} \quad x_3 = z$$

where $\theta \in [0, 2\pi[, \quad z \in [-h, h[ \quad \text{and} \quad r > 0$

In Figure 1, $(e_1, e_2) = \left( \frac{1}{r} \frac{\partial p_0}{\partial \theta}, \frac{\partial p_0}{\partial z} \right)$ denotes the orthonormal basis of the tangent plane at the point $p_0(\theta, z)$.

Note that the curvature operator of $V_0$ is $C_0 = \frac{1}{R} e_1 \otimes e_1$. 
Let us search for shell strain measures having the following form

\[
K = k_1 e_1 \otimes e_1 + k_2 e_2 \otimes e_2 \quad \text{and} \quad U = u_1 e_1 \otimes e_1 + u_2 e_2 \otimes e_2
\]

where \( k_1, k_2, u_1 \) and \( u_2 \) depend only on \( z \). By definition of \( U \), we have \( u_1, u_2 > 0 \).

The strain measures defined before characterize axisymmetric deformations of cylindrical shells. Figure 2 illustrates this kind of deformation.

The aim of the present example is to write compatibility conditions in terms of \( k_1, k_2, u_1 \) and \( u_2 \). To this end, we will use the result obtained in Theorem 2. First, we have

\[
\begin{align*}
&\text{Curl}(U) = -u_1 e_1, \\
&\text{Curl}(KU^{-1}) = -\beta'_2 e_1, \\
&\text{det}(KU^{-1}) = \beta_1 \beta_2, \\
&\text{det}(C_0) = 0, \\
&\text{Adj}(U) = u_1 e_2 \otimes e_2 + u_2 e_1 \otimes e_1 \\
&\text{Adj}(KU^{-1}) = \beta_1 e_2 \otimes e_2 + \beta_2 e_1 \otimes e_1 \quad \text{and} \quad (17)
\end{align*}
\]

where \( \beta_1 = \frac{k_1}{u_1} \), \( \beta_2 = \frac{k_2}{u_2} \) and \( f' = \frac{\partial f}{\partial z} \) for all function \( f = f(z) \).

According to Theorem 2, we get

\[
\Lambda - \frac{U}{\text{det}(U)} \text{Curl}(U) = -v e_1 \quad (18)
\]

where \( v = \frac{u'_1}{u'_2} \).

Therefore, using (17) and (18), the first compatibility condition of Theorem 2 gives

\[-\beta'_1 e_1 = -\beta_2 v e_1\]
and the second condition gives

\[ v' = -\beta_1 \beta_2 \]

Finally, we obtain the following compatibility conditions

\[ \beta_1' - \beta_2 v = 0 \quad \text{and} \quad v' + \beta_1 \beta_2 = 0 \]

which can also take the following form

\[ \beta_1^2 + v^2 = \alpha^2 \quad \text{and} \quad \beta_1' = -\beta_2 v \]

where \( \alpha \) is a real constant.

5. Relation between \( \Lambda \) and the Riemannian curvature

The Riemannian curvature \( R \) of a functional mapping \( \phi : V_0 \mapsto V \) (Breuneval, 1972) is defined as follows.

**Definition 1:** The Riemannian curvature \( R \) is a two-mapping held along \( V_0 \) defined as

\[
R : TV_0 \times TV_0 \rightarrow TV_0
\]

\[ (dp_0, \delta p_0) \mapsto R(dp_0, \delta p_0) = [D(dp_0), D(\delta p_0)] + \dot{d}D(\delta p_0) - \delta D(dp_0) \]

where

\[ D(dp_0) = \left( \frac{\partial p}{\partial p_0} \right)^{-1} \frac{\partial p}{\partial p_0} \]

is the connection of \( \phi \)

and

\[ [D(dp_0), D(\delta p_0)] = D(\delta p_0)D(dp_0) - D(dp_0)D(\delta p_0) \]

The explicit expression of \( R \) is given in the following proposition:

**Proposition 1:** The Riemannian curvature of a functional mapping \( \phi \) is related to \( \Lambda \) as

\[
\mathcal{R}(dp_0, \delta p_0) = -U^{-1} \Sigma_0 U \text{curl}(\Lambda) \sigma_0 (dp_0, \delta p_0)
\]

**Proof** according to the definition of \( D \), we have

\[ D(\delta p_0) = U^{-1}((\Lambda \delta p_0) \Sigma_0 U + \dot{\delta} U) \]

Thus, differentiating \( D \) we get:

\[
\dot{d}D(\delta p_0) = U^{-1}((\Lambda \delta p_0) \Sigma_0 U + \dot{\delta} U) + U^{-1}\Sigma_0 \dot{d}U(\Lambda \delta p_0) + \Sigma_0 U(\dot{\Lambda} \delta p_0) + \frac{\dot{\delta}}{\Sigma} U \]

(20)

Since

\[ D(dp_0)D(\delta p_0) = U^{-1}((\Lambda dp_0) \Sigma_0 U + \dot{d} U)U^{-1}((\Lambda \delta p_0) \Sigma_0 U + \dot{\delta} U) \]

and

\[ U^{-1} \dot{\delta} U U^{-1} = -\dot{\delta} U^{-1} \]
we obtain:

\[ D(dP_0)D(\delta P_0) = -(\bar{\Delta}dP_0)(\bar{\Delta}\delta P_0)I_{TV_0} \]

\[ + U^{-1}\Sigma_0 \delta U(\bar{\Delta}dP_0) - \bar{\Delta}U^{-1}\Sigma_0 U(\bar{\Delta}dP_0) - \bar{\Delta}U^{-1}\delta U \]  

(21)

Adding (20) to (21) we get

\[ D(dP_0)D(\delta P_0) + dD(\delta P_0) = -(\bar{\Delta}dP_0)(\bar{\Delta}\delta P_0)I_{TV_0} + U^{-1}\Sigma_0[\delta \Gamma(\bar{\Delta}dP_0) \]

\[ + \bar{\Delta}U(\bar{\Delta}\delta P_0)] + U^{-1}[\Sigma_0 U(d\delta dP_0) + \Sigma_0 U(\bar{\Delta}d\delta P_0) + \bar{\Delta}\delta U] \]

Finally, using \( d\delta P_0 = \delta dP_0 \) we obtain

\[ R(dP_0, \delta P_0) = -U^{-1}\Sigma_0 U((d\bar{\Delta}\delta P_0) - (\bar{\Delta}d\delta P_0)) = -U^{-1}\Sigma_0 U \text{curl}(\Lambda)\sigma_0(dP_0, \delta P_0) \]

Contrary to Definition 1, result (19) gives an explicit expression of \( R \) in terms of \( U \) and \( \Lambda \). Moreover, using relation (14) we get

\[ R(dP_0, \delta P_0) = -U^{-1}\Sigma_0 U \text{curl}(\frac{U}{\text{det}(U)} \bar{\text{curl}}(U))\sigma_0(dP_0, \delta P_0) \]

which proves that \( R \) depends only on the membrane strain measure.

6. Linearization

In this section, we will linearize the compatibility equations for small deformations and displacements. The small parameters of the linearization are \( \varepsilon = \frac{\gamma^2 - I_{TV_0}}{2} \) (the membrane strain measure) and \( \gamma = K - C_0 \) (the curvature strain measure).

The linearization of \( U, \text{det}(U), KU^{-1}, \text{det}(KU^{-1}) \) leads to

\[ U = I_{TV_0} + \varepsilon + ... \]

\[ \text{det}(U) = 1 - \text{tr}(\varepsilon) + ... \]

\[ U^{-1} = I_{TV_0} - \varepsilon + ... \]

\[ \text{det}(U^{-1}) = 1 + \text{tr}(\varepsilon) + ... \]

\[ KU^{-1} = C_0 + (\gamma - C_0\varepsilon) + ... \]

\[ \text{det}(KU^{-1}) = \text{det}(C_0) + \text{tr}(\text{Adj}(C_0)(\gamma - C_0\varepsilon)) + ... \]

Therefore

\[ \text{curl}(U) = \text{curl}(\varepsilon) + ... \]

\[ \text{curl}(KU^{-1}) = \text{curl}(\gamma - C_0\varepsilon) + ... \]

\[ \Lambda = \text{curl}(\varepsilon) + ... \]

because \( \overline{\text{curl}}(I_{TV_0}) = \overline{\text{curl}}(C_0) = 0 \).

According to the previous results, compatibility equations (12) and (13) become

\[ \text{curl}(\overline{\text{curl}}(\varepsilon)) + \text{tr}(\text{Adj}(C_0)(\gamma - C_0\varepsilon)) = 0 \]  

(22)
and

$$\text{Curl}(\gamma - C_0\varepsilon) = \text{Curl}(\varepsilon)\text{Adj}(C_0)$$

The compatibility equations of linearized strain measures have already been developed in Valid (1976). In this work, the formulation obtained makes use of the divergence operator along the deformed midsurface instead of the curl operator along the reference midsurface. Nevertheless, it is possible after some transformations of Valid's equations, to find Eqs. (22) and (23).

7. Closure

In this paper an intrinsic formulation of compatibility conditions in nonlinear shell theory has been developed. Characterizing the tangential part of the derivative of $R$, we have introduced a new variable $\Lambda$ which simplifies differential calculus along the reference configuration. So, the compatibility equations are naturally decomposed into a first-order differential system. Moreover, the use of $\Lambda$ instead of $\mathcal{R}$ (Breuneval, 1972) makes the compatibility conditions more explicit. On the other hand, thanks to relation (19), our formulation can be compared to certain others [e.g. Breuneval, 1972; Valid, 1973].

In Section 5, we prove that the linearization of compatibility equations leads to the classical results of the linear case (Valid, 1973, 1976).

Finally, it is possible with this formulation to construct the deformed shell midsurface when $U$ and $K$ are known. Let us decompose the determination of the deformed shell midsurface into the following steps: (1) we calculate $\Lambda$ using (14); (2) we insert the expression of $\Lambda$ into Eqs (12) and (13), and verify if $\Delta = U^2$ and $K$ satisfy these equations; (3) if (12) and (13) are satisfied, we integrate (Frobenius method):

$$dR = (\Lambda dp_0)R\Sigma_0 + Ndp_0KU^{-1} + RC_0dp_0N_0$$

(4) having the value of $R$, we can integrate $dp = RUdp_0$ (Frobenius method) and find the deformed shell midsurface.

References

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