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An asymptotic non-linear model for thin-walled rods with strongly curved open cross-section

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Abstract

In this paper, we present a non-linear one-dimensional model for thin-walled rods with open strongly curved cross-section, obtained by asymptotic methods. A dimensional analysis of the non-linear three-dimensional equilibrium equations lets appear dimensionless numbers which reflect the geometry of the structure and the level of applied forces. For a given force level, the order of magnitude of the displacements and the corresponding one-dimensional model are deduced by asymptotic expansions.

Keywords: Thin-walled rod model; Non-linear elasticity; Asymptotic methods

1. Introduction

The interest of thin-walled rods in industry is known since a long time: they provide a maximum of stiffness with a minimum of weight. Whereas in linear elasticity Vlassov model is the one classically used by engineers [1,2], in non-linear elasticity it does not seem to exist any classical model. However, for moderate and large displacements, it is important for engineers to have non-linear models and to know precisely their domain of validity. Such models are necessary to study the buckling and the post-critic behavior of thin-walled rods.[1]

The non-linear models of thin-walled rods existing in the literature are generally deduced with an approach based on a priori assumptions: Vlassov kinematics assumptions are extended to moderate and large rotations [3–8]. The expression of the displacement field so obtained is then introduced in the elastic strain energy of the rod. However, the obtained equilibrium equations are strongly non-linear and coupled. So that supplementary assumptions are generally made to neglect some high-order terms according to the physical phenomenon studied. For an example, to model the "shortening effect" observed for large rotations, Ghobarah and Tso [4] have neglected the non-linear bending terms. They obtained a cubic model with respect to the twist angle Θ. An extension to the previous demarche to profiles with variable sections has been proposed in [6,9]. More recently, Mohri et al. [10] used a general non-linear model[2] to study the "shortening effect" on the post-critic behavior of thin-walled rods. On the other hand, the generalized beam theory developed by Davies and Leach in linear elasticity [11] has been extended to a non-linear behavior [12]. Finally, let us notice other works where a linear Vlassov kinematics is coupled with a non-linear expression of the strain tensor [13,14]. In this case the domain of validity of the obtained model is difficult to specify.

The limitations of these approaches relying on a priori assumptions are twice in the non-linear case. One hand, the Saint-Venant Kirchoff constitutive law used leads to the same contradiction as in the linear case.[3] On the other hand, most

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[1] We recall that linear models enable to access to the critical load but not to the post-critic behavior.

[2] No supplementary assumption is made to neglect a priori non-linear terms in equilibrium equations.

[3] Indeed, the transverse shear E_{t3} is assumed to be equal to zero (this constitutes the non-distortion assumption) whereas the corresponding shear stress Σ_{t3} is different from zero. This is in contradiction with the elastic constitutive law used.

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of the authors make a priori supplementary assumptions to simplify the non-linear equilibrium equations obtained. Therefore it does not seem to exist a classical non-linear model in thin-walled rod theory whose domain of validity can be specify precisely.

In plate and shell theory, there exist many works based on asymptotic methods whose goal is to justify rigorously classical linear and non-linear models [15–21]. In linear theory of beams, the first works on the subject are due to Rigolot [22]. More recently, other justifications of linear and non-linear beam models by asymptotic expansion have been developed in [23–26] in the framework of isotropic elasticity. In the case of anisotropic or heterogeneous linear elastic beams, higher-order approximations based on asymptotic methods have been proposed in [27–29].

In the linear case, an extension of the previous works to thin-walled rods has been proposed in [30,31]. It is based on the asymptotic behavior of Poisson equation in a thin domain when the thickness tends towards zero [32]. Let us cite also [33] where a model for thin-walled beam is deduced from the linear Koiter shell model.

However, in non-linear elasticity, there exists nearly no work concerning the asymptotic justification of non-linear models of thin-walled rods. Let us only cite the generalized finite-element-based beam formulation [34] based on the variational asymptotic method developed by Berdichevsky [35]. This variational asymptotic method has been also applied by Hodges et al. [36] to the geometrically non-linear anisotropic elasticity problem to deduce an anisotropic non-linear thin-walled beam model. The obtained model is composed of a linear two-dimensional problem (describing the strain of the cross-section) and of a non-linear one-dimensional problem characterizing the deformation of the reference line. However, the obtained one-dimensional model is different from the "classical" beam model because of the existence of shear terms in the expression of the strain energy. The results obtained from this model are rather closed to experiments made on prismatic composite thin-walled beams. Finally, let us notice the work of Harursampath and Hodges [37] based on asymptotic expansions but limited to the case of anisotropic tubes.

We propose in this paper to use the constructive approach based on asymptotic expansions, already developed by the authors for plates [38,39] and shells [40,41], to deduce a non-linear model for thin-walled rods from three-dimensional equations. The present work constitutes a generalization of [31,42] in non-linear elasticity. The approach used is based on a decomposition of the three-dimensional equations on Frenet basis of the initial profile. A dimensional analysis then lets appear pertinent dimensionless numbers characterizing the geometry and the level of applied loads. These numbers are measurable and enable to define the domain of validity of the obtained model. Moreover with this approach, the unknowns of the problem (the stresses and the displacements) are directly deduced by asymptotic expansion from the level of applied forces, without any a priori assumptions. This constitutes the constructive character of our approach.

We limit here our analysis to thin-walled rods with strongly curved profile subjected to moderate force levels. In Lemmas 1 and 2, we begin with deducing the order of magnitude of the displacements from the level of applied forces. Then at Result 1, we prove that the displacement field at the leading order has a structure which generalizes Vlassov kinematics in the non-linear case, in particular for moderate and large rotations. At Result 2, the stress field is computed at the leading order of the expansion. Finally, the associated extension, twist and bending equations are deduced at Results 3–5. The equilibrium equations obtained constitute a non-linear system of strongly coupled differential equations, which does not seem to have any equivalent in the literature.

2. The three-dimensional problem

We assume once and for all that an origin O and an orthonormal basis (e_1, e_2, e_3) have been chosen in $\mathbb{R}^3$. We index by a star (*) all dimensional variables and the variables without a star will denote dimensionless variables. On the other hand, within the framework of large displacements, the reference and the current configurations cannot be confused. So that the reference configuration variables will be indexed by $(_0)$. Let ω_0^* be an open cylindrical surface of $\mathbb{R}^3$, (Oe_3) its axis, whose length is L_0 and diameter d_0. We note γ_{0g}^* and γ_{0d}^* its lateral boundary, $\gamma_{01}^* = \omega_0^* \times \{0\}$ and $\gamma_{02}^* = \omega_0^* \times \{L_0\}$ its extremities.

Let us consider now a thin-walled rod with open cross-section and $2h_0$ thickness, whose middle surface is ω_0^*. The thin-walled rod occupies the set $\overline{\Omega}_0^* = \overline{\omega}_0^* \times [-h_0, h_0]$ of $\mathbb{R}^3$ in its reference configuration. We call $\Gamma_{01}^* = \gamma_{01}^* \times]-h_0, h_0[$ and $\Gamma_{02}^* = \gamma_{02}^* \times]-h_0, h_0[$ the extreme faces, $\Gamma_{0g}^* = \gamma_{0g}^* \times]-h_0, h_0[$ and $\Gamma_{0d}^* = \gamma_{0d}^* \times]-h_0, h_0[$ the lateral faces, $\Gamma_{0\pm}^* = \omega_0^* \times \{\pm h_0\}$ the upper and lower faces.

Let M_0^* be a generic point of the beam. We decompose the vector $\overrightarrow{OM_0^*}$ as follows:

$$\overrightarrow{OM_0^*} = x_3^* e_3 + \overrightarrow{G_0^* C_0^*} + \overrightarrow{C_0^* m_0^*} + r^* n, \tag{1}$$

where x_3^* is the coordinate of the current cross-section containing M_0^* on the axis $(0x_3^*)$, G_0^* the point of intersection between the axis $(0x_3^*)$ and the current cross-section, C_0^* an arbitrary chosen point in the plane of the cross-section (see Fig. 1) located by its cartesian coordinates (x_1^{c*}, x_2^{c*}), and r^* the thickness variable. We call $\mathscr{C}_0^*$ the intersection curve between ω_0^* and the cross-section. The orthogonal projection m_0^* of M_0^* on the middle surface is located by its cartesian coordinates $x^* = (x_1^*, x_2^*)$ or by its curvilinear abscisse s^* along $\mathscr{C}_0^*$. The origin s_0^* of the curvilinear abscisse is an arbitrary chosen point of $\mathscr{C}_0^*$. We note n the unit normal and t the unit tangent vector of $\mathscr{C}_0^*$. Moreover, we call l^* and q^* the coordinates of the vector $\overrightarrow{C_0^* m_0^*}$ in the basis (t, n). Finally, we call α^* the angle (e_1, t) and c_0^* the curvature of the curve $\mathscr{C}_0^*$ (see Fig. 1).

In what follows, we consider only thin-walled rods such as $d_0/L_0 \ll 1$, $h_0/d_0 \ll 1$ and $h_0 \|c_0^*\|_\infty \ll 1$. We assume that the rod is subjected to the applied body forces $f^* = f_t^* t + f_n^* n + f_3^* e_3$:

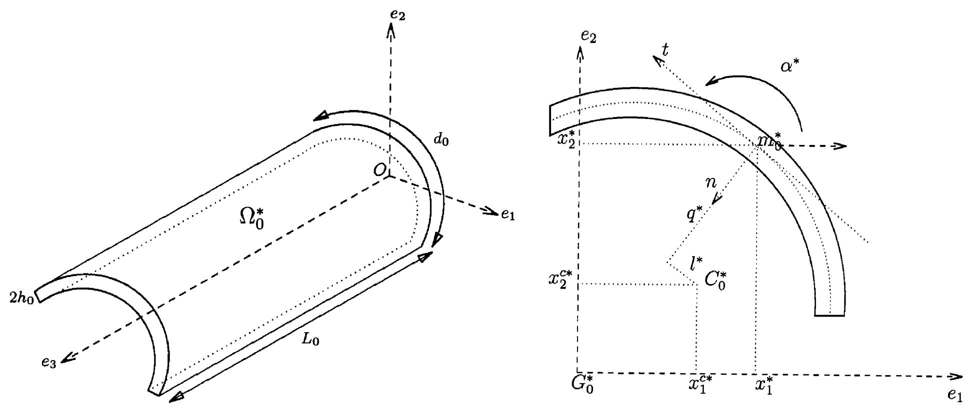

Fig. 1. Geometry of the rod.

$\bar{\Omega}_0^* \to \mathbb{R}^3$ and to the applied surface forces $g^{*\pm} = g_t^{*\pm} t + g_n^{*\pm} n + g_3^{*\pm} e_3 \colon \bar{\Gamma}_{0\pm}^* \to \mathbb{R}^3$. Moreover, the rod is assumed to be clamped on its extremities Γ_{01}^* and Γ_{02}^*, and free on its lateral faces Γ_{0g}^* and Γ_{0d}^*. The unknown of the problem is then the displacement $U^* : \bar{\Omega}_0^* \to \mathbb{R}^3$. Within the framework of non-linear elasticity, the displacement U^* and the second Piola–Kirchoff tensor Σ^* satisfy the non-linear equilibrium equations:

$$\begin{cases} \mathrm{Div}^*(\Sigma^* \overline{F}^*) = -f^* & \text{in } \Omega_0^*, \\ (F^*\Sigma^*)\tilde{n} = g^{*\pm} & \text{on } \Gamma_{0\pm}^*, \\ (F^*\Sigma^*)\tilde{t} = 0 & \text{on } \Gamma_{0g}^* \cup \Gamma_{0d}^*, \\ U^* = 0 & \text{on } \Gamma_{01}^* \cup \Gamma_{02}^*, \end{cases} \tag{2}$$

where the overbar denotes the transposition operator, $F^* = \partial\psi^*/\partial M_0^* = I + \partial U^*/\partial M_0^*$ the linear tangent map to the mapping function $M_0^* \to \psi^*(M_0^*) = M_0^* + U^*$, and $\tilde{n}$ (respectively $\tilde{t}$) the external unit normal to the upper and lower faces (respectively to the lateral faces). Limiting our study to Hookean materials, the constitutive relation is given by $\Sigma^* = \lambda \,\mathrm{Tr}\, E^* I + 2\mu E^*$ where $E^* = \frac{1}{2}(\overline{F}^* F^* - I)$ denotes the Green–Lagrange tensor and I the $\mathbb{R}^3$ identity tensor. As in linear elasticity [42], the boundary conditions on $\Gamma_{0g}^* \cup \Gamma_{0d}^*$ are considered on average upon the thickness, in order the twist to be of the same order as the bending in the asymptotic model of Proposition 1.

Finally let us add the mass conservation law to these equilibrium equations. It writes

$$\rho^* \det F^* = \rho_0^*, \tag{3}$$

where ρ_0^* and ρ^* denote the density of the material in the initial and deformed configuration, respectively. We only assume that the density ρ^* of the material does not become infinity or equivalently that

$$\det F^* = \det\left(\frac{\partial\psi^*}{\partial M_0^*}\right) \geqslant a > 0 \quad \text{in } \Omega_0^*, \tag{4}$$

where a is a positive constant. This condition will be used later.

3. Dimensional analysis of equilibrium equations

3.1. Decomposition of equations

Let us decompose the three-dimensional equations in Frenet basis (t, n, e_3) of the initial configuration. To do this, let us decompose the mapping ψ^* as follows:

$$\psi^* = \psi_t^* t + \psi_n^* n + \psi_3^* e_3. \tag{5}$$

Then F^* can be written in the basis (t, n, e_3) as follows:

$$F^* = \frac{\partial\psi^*}{\partial M_0^*} = \begin{bmatrix} k_0^*\left(\dfrac{\partial\psi_t^*}{\partial s^*} - c_0^*\psi_n^*\right) & \dfrac{\partial\psi_t^*}{\partial r^*} & \dfrac{\partial\psi_t^*}{\partial x_3^*} \\ k_0^*\left(\dfrac{\partial\psi_n^*}{\partial s^*} + c_0^*\psi_t^*\right) & \dfrac{\partial\psi_n^*}{\partial r^*} & \dfrac{\partial\psi_n^*}{\partial x_3^*} \\ k_0^*\dfrac{\partial\psi_3^*}{\partial s^*} & \dfrac{\partial\psi_3^*}{\partial r^*} & \dfrac{\partial\psi_3^*}{\partial x_3^*} \end{bmatrix}, \tag{6}$$

where we set

$$k_0^* = \frac{1}{1 - r^* c0^*}. \tag{7}$$

It is also possible to write F^* on the more compact form

$$F^* = \left[k_0^* \frac{\partial\psi^*}{\partial s^*} \; \frac{\partial\psi^*}{\partial r^*} \; \frac{\partial\psi^*}{\partial x_3^*} \right]. \tag{8}$$

Let us notice that the thin-walled rod assumption ($h_0\|c_0\|_\infty \ll 1$) ensures the existence of k_0^*.

Then using decomposition (8) of F^*, the Green–Lagrange strain tensor writes E^*:

$$2E^* = \begin{bmatrix} k_0^{*2}\|\dfrac{\partial\psi^*}{\partial s^*}\|^2 - 1 & k_0^*\overline{\dfrac{\partial\psi^*}{\partial r^*}}\dfrac{\partial\psi^*}{\partial s^*} & k_0^*\overline{\dfrac{\partial\psi^*}{\partial x_3^*}}\dfrac{\partial\psi^*}{\partial s^*} \\ k_0^*\overline{\dfrac{\partial\psi^*}{\partial s^*}}\dfrac{\partial\psi^*}{\partial r^*} & \left\|\dfrac{\partial\psi^*}{\partial r^*}\right\|^2 - 1 & \overline{\dfrac{\partial\psi^*}{\partial x_3^*}}\dfrac{\partial\psi^*}{\partial r^*} \\ k_0^*\overline{\dfrac{\partial\psi^*}{\partial s^*}}\dfrac{\partial\psi^*}{\partial x_3^*} & \overline{\dfrac{\partial\psi^*}{\partial r^*}}\dfrac{\partial\psi^*}{\partial x_3^*} & \|\dfrac{\partial\psi^*}{\partial x_3^*}\|^2 - 1 \end{bmatrix}. \tag{9}$$

In the same way, the Piola–Kirchoff stress tensor Σ^* can be decomposed in the (t, n, e_3) basis as follows:

$$\Sigma^* = \begin{bmatrix} \sigma^*_{tt} & \sigma^*_{tn} & \sigma^*_{t3} \\ \sigma^*_{tn} & \sigma^*_{nn} & \sigma^*_{n3} \\ \sigma^*_{t3} & \sigma^*_{n3} & \sigma^*_{33} \end{bmatrix} \tag{10}$$

with

$$\sigma^*_{tt} = \frac{\lambda+2\mu}{2} k_0^{*2} \left\| \frac{\partial\psi^*}{\partial s^*} \right\|^2 + \frac{\lambda}{2} \left\| \frac{\partial\psi^*}{\partial r^*} \right\|^2 + \frac{\lambda}{2} \left\| \frac{\partial\psi^*}{\partial x_3^*} \right\|^2 - \frac{3\lambda+2\mu}{2} \qquad \sigma^*_{tn} = \mu k_0^* \overline{\frac{\partial\psi^*}{\partial r^*}} \frac{\partial\psi^*}{\partial s^*},$$

$$\sigma^*_{nn} = \frac{\lambda}{2} k_0^{*2} \left\| \frac{\partial\psi^*}{\partial s^*} \right\|^2 + \frac{\lambda+2\mu}{2} \left\| \frac{\partial\psi^*}{\partial r^*} \right\|^2 + \frac{\lambda}{2} \left\| \frac{\partial\psi^*}{\partial x_3^*} \right\|^2 - \frac{3\lambda+2\mu}{2} \qquad \sigma^*_{t3} = \mu k_0^* \overline{\frac{\partial\psi^*}{\partial x_3^*}} \frac{\partial\psi^*}{\partial s^*},$$

$$\sigma^*_{33} = \frac{\lambda}{2} k_0^{*2} \| \frac{\partial\psi^*}{\partial s^*} \|^2 + \frac{\lambda}{2} \| \frac{\partial\psi^*}{\partial r^*} \|^2 + \frac{\lambda+2\mu}{2} \| \frac{\partial\psi^*}{\partial x_3^*} \|^2 - \frac{3\lambda+2\mu}{2} \quad \sigma^*_{n3} = \mu \overline{\frac{\partial\psi^*}{\partial x_3^*}} \frac{\partial\psi^*}{\partial r^*}.$$

To simplify the notations in what follows, we will write F^* in the following form:

$$F^* = \begin{bmatrix} k_0^* \frac{\partial\psi^*}{\partial s^*} \cdot t & \frac{\partial\psi^*}{\partial r^*} \cdot t & \frac{\partial\psi^*}{\partial x_3^*} \cdot t \\ k_0^* \frac{\partial\psi^*}{\partial s^*} \cdot n & \frac{\partial\psi^*}{\partial r^*} \cdot n & \frac{\partial\psi^*}{\partial x_3^*} \cdot n \\ k_0^* \frac{\partial\psi^*}{\partial s^*} \cdot e_3 & \frac{\partial\psi^*}{\partial r^*} \cdot e_3 & \frac{\partial\psi^*}{\partial x_3^*} \cdot e_3 \end{bmatrix}. \tag{11}$$

Then setting $\tau^* = \Sigma^* \overline{F^*}$, we have

$$\tau^* = \Sigma^* \overline{F^*} = \begin{bmatrix} \tau^*_{tt} & \tau^*_{tn} & \tau^*_{t3} \\ \tau^*_{nt} & \tau^*_{nn} & \tau^*_{n3} \\ \tau^*_{3t} & \tau^*_{3n} & \tau^*_{33} \end{bmatrix} = \begin{bmatrix} \overline{\tau}^*_t \\ \overline{\tau}^*_n \\ \overline{\tau}^*_3 \end{bmatrix}, \tag{12}$$

where the components of the tensor τ^* are given by

$$\tau^*_{tt} = \overline{\tau^*_t} t \quad \tau^*_{tn} = \overline{\tau^*_t} n \quad \tau^*_{t3} = \overline{\tau^*_t} e_3,$$

$$\tau^*_{nt} = \overline{\tau^*_n} t \quad \tau^*_{nn} = \overline{\tau^*_n} n \quad \tau^*_{n3} = \overline{\tau^*_n} e_3,$$

$$\tau^*_{3t} = \overline{\tau^*_3} t \quad \tau^*_{3n} = \overline{\tau^*_3} n \quad \tau^*_{33} = \overline{\tau^*_3} e_3.$$

We recall that the overbar denotes the transposition operator. Thus if u and v are two vectors of $\mathbb{R}^3$, we have $\overline{u}v = u.v$ where the dot denotes the scalar product of $\mathbb{R}^3$. Equivalently, the vectors $\overline{\tau}^*_t$, $\overline{\tau}^*_n$ and $\overline{\tau}^*_3$ can be written with respect to the mapping ψ^*. We have

$$\tau^*_t = k_0^* \left\{ \frac{\lambda+2\mu}{2} k_0^{*2} \left\| \frac{\partial\psi^*}{\partial s^*} \right\|^2 + \frac{\lambda}{2} \left\| \frac{\partial\psi^*}{\partial r^*} \right\|^2 + \frac{\lambda}{2} \left\| \frac{\partial\psi^*}{\partial x_3} \right\|^2 - \frac{3\lambda+2\mu}{2} \right\} \frac{\partial\psi^*}{\partial s} + \mu k_0^* \overline{\frac{\partial\psi^*}{\partial r^*}} \frac{\partial\psi^*}{\partial s^*} \frac{\partial\psi^*}{\partial r} + \mu k_0^* \overline{\frac{\partial\psi^*}{\partial x_3^*}} \frac{\partial\psi^*}{\partial s^*} \frac{\partial\psi^*}{\partial x_3}, \tag{13}$$

$$\tau^*_n = \mu k_0^{*2} \overline{\frac{\partial\psi^*}{\partial r^*}} \frac{\partial\psi^*}{\partial s^*} \frac{\partial\psi^*}{\partial s} + \left\{ \frac{\lambda}{2} k_0^{*2} \left\| \frac{\partial\psi^*}{\partial s^*} \right\|^2 + \frac{\lambda+2\mu}{2} \left\| \frac{\partial\psi^*}{\partial r^*} \right\|^2 + \frac{\lambda}{2} \left\| \frac{\partial\psi^*}{\partial x_3^*} \right\|^2 - \frac{3\lambda+2\mu}{2} \right\} \frac{\partial\psi^*}{\partial r} + \mu \overline{\frac{\partial\psi^*}{\partial x_3^*}} \frac{\partial\psi^*}{\partial r^*} \frac{\partial\psi^*}{\partial x_3}, \tag{14}$$

$$\tau^*_3 = \mu k_0^{*2} \overline{\frac{\partial\psi^*}{\partial x_3^*}} \frac{\partial\psi^*}{\partial s^*} \frac{\partial\psi^*}{\partial s} + \mu \overline{\frac{\partial\psi^*}{\partial r^*}} \frac{\partial\psi^*}{\partial x_3^*} \frac{\partial\psi^*}{\partial r} + \left\{ \frac{\lambda}{2} k_0^{*2} \left\| \frac{\partial\psi^*}{\partial s^*} \right\|^2 + \frac{\lambda}{2} \left\| \frac{\partial\psi^*}{\partial r^*} \right\|^2 + \frac{\lambda+2\mu}{2} \left\| \frac{\partial\psi^*}{\partial x_3^*} \right\|^2 - \frac{3\lambda+2\mu}{2} \right\} \frac{\partial\psi^*}{\partial x_3}. \tag{15}$$

They can also be written with respect to the components of the stresses Σ^* as follows:

$$\tau^*_t = k_0^* \sigma^*_{tt} \frac{\partial\psi^*}{\partial s^*} + \sigma^*_{tn} \frac{\partial\psi^*}{\partial r^*} + \sigma^*_{t3} \frac{\partial\psi^*}{\partial x_3^*}, \tag{16}$$

$$\tau^*_n = k_0^* \sigma^*_{nt} \frac{\partial\psi^*}{\partial s^*} + \sigma^*_{nn} \frac{\partial\psi^*}{\partial r^*} + \sigma^*_{n3} \frac{\partial\psi^*}{\partial x_3^*}, \tag{17}$$

$$\tau^*_3 = k_0^* \sigma^*_{3t} \frac{\partial\psi^*}{\partial s^*} + \sigma^*_{3n} \frac{\partial\psi^*}{\partial r^*} + \sigma^*_{33} \frac{\partial\psi^*}{\partial x_3^*}. \tag{18}$$

Now the three-dimensional equilibrium equations can be decomposed in Frenet basis (t, n, e_3). To do this we use the decomposition of the three-dimensional divergence. We obtain in Ω_0^*:

$$\frac{\partial\tau^*_{nt}}{\partial r^*} + k_0^* \left(\frac{\partial\tau^*_{tt}}{\partial s^*} - c_0^* \tau^*_{tn} - c_0^* \tau^*_{nt} \right) + \frac{\partial\tau^*_{3t}}{\partial x_3^*} = -f^*_t, \tag{19}$$

$$\frac{\partial\tau^*_{nn}}{\partial r^*} + k_0^* \left(\frac{\partial\tau^*_{tn}}{\partial s^*} + c_0^* \tau^*_{tt} - c_0^* \tau^*_{nn} \right) + \frac{\partial\tau^*_{3n}}{\partial x_3^*} = -f^*_n, \tag{20}$$

$$\frac{\partial\tau^*_{n3}}{\partial r^*} + k_0^* \left(\frac{\partial\tau^*_{t3}}{\partial s^*} - c_0^* \tau^*_{n3} \right) + \frac{\partial\tau^*_{33}}{\partial x_3^*} = -f^*_3 \tag{21}$$

or equivalently

$$\frac{\partial\tau^*_n}{\partial r^*} + k_0^* \left(\frac{\partial\tau^*_t}{\partial s^*} - c_0^* \tau^*_n \right) + \frac{\partial\tau^*_3}{\partial x_3^*} = -f^* \tag{22}$$

with the associated boundary conditions

$$\tau_n^* = g^{*\pm} \quad \text{for } r^* = \pm h_0^* \tag{23}$$

$$\tau_t^* = 0 \quad \text{for } s^* = s_-^* \text{ and } s^* = s_+^* \tag{24}$$

$$U^* = 0 \quad \text{for } x_3^* = 0 \text{ and } x_3^* = L_0. \tag{25}$$

3.2. Dimensional analysis of equations

Let us define the following dimensionless physical data and dimensionless unknowns of the problem:

$$u_t = \frac{u_t^*}{u_{tr}} \quad u_n = \frac{u_n^*}{u_{nr}} \quad u_3 = \frac{u_3^*}{u_{3r}} \quad x_3 = \frac{x_3^*}{L_0} \quad x = \frac{x^*}{d_0},$$

$$s = \frac{s^*}{d_0} \quad r = \frac{r^*}{h_0} \quad c_0 = \frac{c_0^*}{c_r} \quad f_t = \frac{f_t^*}{f_{tr}} \quad f_n = \frac{f_n^*}{f_{nr}},$$

$$f_3 = \frac{f_3^*}{f_{3r}} \quad g_t = \frac{g_t^*}{g_{tr}} \quad g_n = \frac{g_n^*}{g_{nr}} \quad g_3 = \frac{g_3^*}{g_{3r}} \quad \rho = \frac{\rho^*}{\rho_0^*},$$

where the variables indexed by $(_r)$ are the reference ones. The new variables which appear (without a star) are dimensionless. To avoid any assumption on the order of magnitude of the displacement components, the reference scales u_{tr}, u_{nr} and u_{3r} are firstly assumed to be equal to L_0. This a priori allows large displacements.

In a natural way we introduce $c_r = \|c_0^*\|_\infty$ which denotes the maximum of curvature (the smaller radius of curvature) of the middle surface ω_0^*. As in shell theory, the order of magnitude of the curvature is a fundamental data in the asymptotic expansion of equations. Therefore, we will have to distinguish the rods with shallow cross profile from the rods with strongly curved profile.

First the dimensional analysis of k_0^* leads to

$$k_0 = \frac{1}{1 - h_0 c_r r c_0}.$$

If we set $v = hc_r$, the thin-walled rod assumption ensures that $v \ll 1$ and so that k_0 admits the following expansion:

$$k_0 = 1 + vrc_0 + (vrc_0)^2 + \cdots. \tag{26}$$

In a natural way, we have

$$\psi^* = L_0 \psi$$

whose components can be written as

$$\frac{\partial \psi^*}{\partial r^*} = \varepsilon^{-1}\eta^{-1}\frac{\partial \psi}{\partial r} \quad \frac{\partial \psi^*}{\partial s^*} = \varepsilon^{-1}\frac{\partial \psi}{\partial s} \quad \frac{\partial \psi^*}{\partial x_3^*} = \frac{\partial \psi}{\partial x_3},$$

where we set

$$\varepsilon = \frac{d_0}{L_0} \quad \text{and} \quad \eta = \frac{h_0}{d_0}.$$

Using (13), (14) and (15), the dimensional analysis of τ_t^*, τ_n^* and τ_3^* leads to

$$\begin{aligned}\tau_t = \varepsilon^{-1} k_0 \Bigg\{ & \varepsilon^{-2}\frac{\beta+2}{2}k_0^2\left\|\frac{\partial\psi}{\partial s}\right\|^2 + \varepsilon^{-2}\eta^{-2}\frac{\beta}{2}\left\|\frac{\partial\psi}{\partial r}\right\|^2 \\ & + \frac{\beta}{2}\left\|\frac{\partial\psi}{\partial x_3}\right\|^2 - \frac{3\beta+2}{2}\Bigg\}\frac{\partial\psi}{\partial s} \\ & + \varepsilon^{-3}\eta^{-2}k_0\overline{\frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial s}}\frac{\partial\psi}{\partial r} + \varepsilon^{-1}k_0\overline{\frac{\partial\psi}{\partial x_3}\frac{\partial\psi}{\partial s}}\frac{\partial\psi}{\partial x_3},\end{aligned} \tag{27}$$

$$\begin{aligned}\tau_n = \varepsilon^{-1}\eta^{-1} \Bigg\{ & \varepsilon^{-2}k_0^2\frac{\beta}{2}\left\|\frac{\partial\psi}{\partial s}\right\|^2 + \varepsilon^{-2}\eta^{-2}\frac{\beta+2}{2} \\ & \times \left\|\frac{\partial\psi}{\partial r}\right\|^2 + \frac{\beta}{2}\left\|\frac{\partial\psi}{\partial x_3}\right\|^2 - \frac{3\beta+2}{2}\Bigg\}\frac{\partial\psi}{\partial r} \\ & + k_0^2\varepsilon^{-3}\eta^{-1}\overline{\frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial s}}\frac{\partial\psi}{\partial s} + \varepsilon^{-1}\eta^{-1}\overline{\frac{\partial\psi}{\partial x_3}\frac{\partial\psi}{\partial r}}\frac{\partial\psi}{\partial x_3},\end{aligned} \tag{28}$$

$$\begin{aligned}\tau_3 = \Bigg\{ & \varepsilon^{-2}\frac{\beta}{2}k_0^2\left\|\frac{\partial\psi}{\partial s}\right\|^2 + \varepsilon^{-2}\eta^{-2}\frac{\beta}{2}\left\|\frac{\partial\psi}{\partial r}\right\|^2 \\ & + \frac{\beta+2}{2}\left\|\frac{\partial\psi}{\partial x_3}\right\|^2 - \frac{3\beta+2}{2}\Bigg\}\frac{\partial\psi}{\partial x_3} \\ & + k_0^2\varepsilon^{-2}\overline{\frac{\partial\psi}{\partial x_3}\frac{\partial\psi}{\partial s}}\frac{\partial\psi}{\partial s} + \varepsilon^{-2}\eta^{-2}\overline{\frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial x_3}}\frac{\partial\psi}{\partial r},\end{aligned} \tag{29}$$

where we set $\beta = \lambda/\mu$ and $\tau = \tau^*/\mu$. Equivalently, using (16), (17) and (18), we get

$$\tau_t = \varepsilon^{-1}k_0\sigma_{tt}\frac{\partial\psi}{\partial s} + \varepsilon^{-2}\sigma_{tn}\frac{\partial\psi}{\partial r} + \sigma_{t3}\frac{\partial\psi}{\partial x_3}, \tag{30}$$

$$\tau_n = \varepsilon^{-1}k_0\sigma_{nt}\frac{\partial\psi}{\partial s} + \varepsilon^{-2}\sigma_{nn}\frac{\partial\psi}{\partial r} + \sigma_{n3}\frac{\partial\psi}{\partial x_3}, \tag{31}$$

$$\tau_3 = \varepsilon^{-1}k_0\sigma_{3t}\frac{\partial\psi}{\partial s} + \varepsilon^{-2}\sigma_{3n}\frac{\partial\psi}{\partial r} + \sigma_{33}\frac{\partial\psi}{\partial x_3}. \tag{32}$$

Finally the non-dimensional three-dimensional equilibrium equations can be written on the compact form

$$\frac{\partial\tau_n}{\partial r} + k_0\left(\eta\frac{\partial\tau_t}{\partial s} - vc_0\tau_n\right) + \varepsilon\eta\frac{\partial\tau_3}{\partial x_3} = -\mathscr{F}f \tag{33}$$

with the associated boundary conditions

$$\tau_n = \mathscr{G}g^{\pm} \quad \text{for } r = \pm 1, \tag{34}$$

$$\tau_t = 0 \quad \text{for } s = s_- \text{ and } s = s_+, \tag{35}$$

$$\tau_3 = 0 \quad \text{for } x_3 = 0 \text{ and } x_3 = 1. \tag{36}$$

The mass conservation law (4) becomes

$$\rho \det(F) = \rho \det\left(\frac{\partial\psi}{\partial M_0}\right) \geqslant \varepsilon^3 a > 0 \quad \forall \varepsilon > 0. \tag{37}$$

Therefore the dimensional analysis of equilibrium equations naturally lets appear non-dimensional numbers

$$\varepsilon = \frac{d_0}{L_0} \quad \eta = \frac{h_0}{d_0} \quad v = h_0 c_r \quad \mathscr{F} = \frac{h_0 f_r}{\mu} \quad \mathscr{G} = \frac{g_r}{\mu}.$$

(1) The shape ratio ε characterizes the inverse of the shooting-pain of the rod. This is a known parameter of the problem which satisfies $\varepsilon \ll 1$.
(2) The dimensional number η denotes the ratio between the thickness h_0 of the rod to the length of its profile. This number is also a data of the problem which satisfies $\eta \ll 1$.
(3) The shape ratio $v = h_0 c_r$ is the ratio between the thickness to the smaller radius of curvature of the middle surface ω_0 of the rod. It is a given geometrical data of the problem which verifies $v \ll 1$.
(4) The force ratios $\mathscr{F}$ and $\mathscr{G}$ represent, respectively, the ratio of the resultant on the thickness of the body forces (respectively of the surface forces) to μ considered as a reference stress. These numbers only depend on known physical quantities and must be considered as given data of the problem.

3.3. One-scale assumption

To reduce the problem to a one-scale problem, ε is chosen as the small reference parameter of the problem.[4] Then the other dimensional numbers are linked to ε, or more precisely to the powers of ε.

- *Order of magnitude of the curvature*
  In a natural way, as in shell theory, we have to distinguish thin-walled rods:
  - with strongly curved profile where $v = \varepsilon$
  - with shallow profile where $v = \varepsilon^2$.
    This distinction is fundamental because these two families of thin-walled rods do not have the same asymptotic behavior.
- *Order of magnitude of the thickness*
  Three cases can be distinguished and studied:
  - the thick rods where $\eta = 1$. This is not the subject of this paper.
  - the thin-walled rods where $\eta = \varepsilon$. It is the case studied here.
  - the very thin-walled rods where $\eta = \varepsilon^p$, $p > 1$. This case is not studied in this paper.
- *Order of magnitude of the applied loads*
  The applied loads are an essential given data of the problem. In the framework of a one-scale asymptotic expansion, the force ratios must be linked also to ε. This is equivalent to fix the order of magnitude of the applied forces which are given data.

[4] If not we have multi-scale problems which are much more complicated. It is not the subject of this paper.

In the case of thin-walled rods with strongly curved profile, we will consider moderate levels of applied forces such as $\mathscr{F}_t = \mathscr{F}_n = \varepsilon^5$, $\mathscr{G}_t = \mathscr{G}_n = \varepsilon^5$ and $\mathscr{F}_3 = \mathscr{G}_3 = \varepsilon^4$.

4. Asymptotic expansion of equations

Let us consider a thin-walled rod with strongly curved profile ($v = \varepsilon$) subjected to force levels such as $\mathscr{F}_t = \mathscr{F}_n = \varepsilon^5$, $\mathscr{G}_t = \mathscr{G}_n = \varepsilon^5$ and $\mathscr{F}_3 = \mathscr{G}_3 = \varepsilon^4$. The problem is then reduced to a one scale one which writes in Ω_0:

$$\frac{\partial \tau_n}{\partial r} + \varepsilon k_0 \left(\frac{\partial \tau_t}{\partial s} - c_0 \tau_n \right) + \varepsilon^2 \frac{\partial \tau_3}{\partial x_3} = -\mathscr{F} f$$

with the boundary conditions

$$\tau_n = \mathscr{G} g^{\pm} \quad \text{for } r = \pm 1,$$

$$\tau_t = 0 \quad \text{for } s = s_- \text{ and } s = s_+,$$

$$U = 0 \quad \text{for } x_3 = 0 \text{ and } 1,$$

where $\mathscr{F}$ and $\mathscr{G}$ are two diagonal matrix defined as follows:

$$\mathscr{F} = \begin{bmatrix} \mathscr{F}_t & 0 & 0 \\ 0 & \mathscr{F}_n & 0 \\ 0 & 0 & \mathscr{F}_3 \end{bmatrix}, \quad \mathscr{G} = \begin{bmatrix} \mathscr{G}_t & 0 & 0 \\ 0 & \mathscr{G}_n & 0 \\ 0 & 0 & \mathscr{G}_3 \end{bmatrix}.$$

We recall that we have to relax the boundary condition on the free lateral surface for $s = s_-$ and $s = s_+$ to avoid boundary layers. Thus we have in particular

$$\int_{-1}^{1} \tau_{t3} \, \mathrm{d}r = 0 \quad \text{for } s = s_{\pm}.$$

The standard asymptotic technique then proceeds as follows. First we postulate that the solution $U = (u_t, u_n, u_3)$ of the problem admits a formal expansion with respect to the powers of ε:

$$U = U^0 + \varepsilon^1 U^1 + \varepsilon^2 U^2 + \cdots.$$

The expansion of U with respect to ε implies an expansion of the components of the mapping ψ, of the strain tensor E, and of the stresses Σ and $\tau = \Sigma \bar{F}$ with respect to ε:

$$\psi = \psi^0 + \varepsilon \psi^1 + \varepsilon^2 \psi^2 + \cdots,$$

$$E = E^0 + \varepsilon E^1 + \varepsilon^2 E^2 + \cdots,$$

$$\Sigma = \Sigma^0 + \varepsilon \Sigma^1 + \varepsilon^2 \Sigma^2 + \cdots,$$

$$\tau = \tau^0 + \varepsilon \tau^1 + \varepsilon^2 \tau^2 + \cdots.$$

Then we replace ψ by its expansion in equilibrium equations and we equate to zero the factor of the successive powers of ε. This way we obtain a succession of coupled problems $\mathscr{P}_{-6}$, $\mathscr{P}_{-5}$, $\mathscr{P}_{-4}, \ldots$ whose resolution leads to the search asymptotic one-dimensional model corresponding to the force level considered.

It is important to notice that with the approach developed here, the order of magnitude of the displacements (which are unknowns of the problem) are directly deduced from the level of applied forces. In particular, for the force levels considered here, the axial displacement is one order smaller than the other ones. This is the result of Lemmas 1 and 2:

Lemma 1. *For applied force levels such as* $\mathscr{F}_t = \mathscr{F}_n = \varepsilon^5$, $\mathscr{G}_t = \mathscr{G}_n = \varepsilon^5$ *and* $\mathscr{F}_3 = \mathscr{G}_3 = \varepsilon^4$, *the leading term* U^0 *of the expansion of* U *is equal to zero*:

$$U^0 = 0.$$

Proof. The proof of this lemma is split into five steps.

(i) ψ^0 *does not depend on* r

The cancellation of the factor of ε^{-6} leads to problem $\mathscr{P}_{-6}$ which writes

$$\frac{\partial \tau_n^{-6}}{\partial r} = 0 \quad \text{in } \Omega_0,$$

$$\tau_n^{-6} = 0 \quad \text{for } r = \pm 1,$$

$$\tau_t^{-6} = 0 \quad \text{for } s = s_- \text{ and } s = s_+.$$

Thus in Ω_0 we have

$$\tau_n^{-6} = 0.$$

Then replacing τ_n^{-6} with its expression, we get

$$\frac{\beta+2}{2}\left\|\frac{\partial \psi^0}{\partial r}\right\|^2 \frac{\partial \psi^0}{\partial r} = 0.$$

So we have

$$\frac{\partial \psi^0}{\partial r} = 0$$

which implies that

$$\tau_t^{-5} = \tau_t^{-4} = 0 \quad \tau_n^{-6} = \tau_n^{-5} = \tau_n^{-4} = 0$$
$$\tau_3^{-4} = \tau_3^{-3} = 0.$$

Then problems $\mathscr{P}_{-5}$ and $\mathscr{P}_{-4}$ are trivially satisfied.

(ii) ψ^1 *does not depend on* r

The cancellation of the factor of ε^{-3} leads to problem $\mathscr{P}_{-3}$ which reduces to

$$\frac{\partial \tau_n^{-3}}{\partial r} = 0 \quad \text{in } \Omega_0,$$
$$\tau_n^{-3} = 0 \quad \text{for } r = \pm 1,$$
$$\tau_t^{-3} = 0 \quad \text{for } s = s_- \text{ and } s = s_+ \tag{38}$$

and leads, in Ω_0, to

$$\tau_n^{-3} = 0.$$

Then replacing τ_n^{-3} with its expression, we get

$$\left(\overline{\frac{\partial \psi^1}{\partial r}}\,\frac{\partial \psi^0}{\partial s}\right)\frac{\partial \psi^0}{\partial s} + \left(\frac{\beta}{2}\left\|\frac{\partial \psi^0}{\partial s}\right\|^2 + \frac{\beta+2}{2}\left\|\frac{\partial \psi^1}{\partial r}\right\|^2\right)\frac{\partial \psi^1}{\partial r} = 0. \tag{39}$$

Two cases are possible:

- either $\partial\psi^0/\partial s = 0$, then Eq. (39) reduces to

  $$\frac{\beta+2}{2}\left\|\frac{\partial \psi^1}{\partial r}\right\|^2 \frac{\partial \psi^1}{\partial r} = 0$$

  and we have $\partial\psi^1/\partial r = 0$
- else $\partial\psi^0/\partial s \neq 0$, then there exists $\xi \in \mathbb{R}$ such as $\partial\psi^1/\partial r = \xi(\partial\psi^0/\partial s)$. According to (39), we deduce that ξ satisfies

  $$\xi(1+\xi^2) = 0.$$

  Then we have $\xi = 0$, which implies that $\partial\psi^1/\partial r = 0$.

Finally (27) leads to

$$\tau_t^{-3} = \frac{\beta+2}{2}\left\|\frac{\partial \psi^0}{\partial s}\right\|^2 \frac{\partial \psi^0}{\partial s}. \tag{40}$$

(iii) ψ^0 *only depends on* x_3

The cancellation of the factor of ε^{-2} leads to problem $\mathscr{P}_{-2}$ which reduces to

$$\frac{\partial \tau_n^{-2}}{\partial r} + \frac{\partial \tau_t^{-3}}{\partial s} = 0 \quad \text{in } \Omega_0, \tag{41}$$

$$\tau_n^{-2} = 0 \quad \text{for } r = \pm 1, \tag{42}$$

$$\tau_t^{-2} = 0 \quad \text{for } s = s_- \text{ and } s = s_+. \tag{43}$$

Let us integrate Eq. (41) upon the thickness. Using the boundary condition (42), we get

$$\int_{-1}^{1} \frac{\partial \tau_t^{-3}}{\partial s}\, \mathrm{d}r = 0.$$

As τ_t^{-3} does not depend on r, according to (38), we deduce that

$$\tau_t^{-3} = 0$$

which leads to $\partial\psi^0/\partial s = 0$. Thus ψ^0 only depends on x_3.

According to the previous results, we have

$$\tau_t^{-3} = \tau_t^{-2} = \tau_t^{-1} = 0, \quad \tau_n^{-2} = \tau_n^{-1} = 0,$$
$$\tau_3^{-3} = \tau_3^{-2} = \tau_3^{-1} = 0$$

and problems $\mathscr{P}_{-2}$ and $\mathscr{P}_{-1}$ are trivially satisfied.

(iv) $(\partial\psi^1/\partial s, \partial\psi^2/\partial r, \mathrm{d}\psi^0/\mathrm{d}x_3)$ *constitute a basis of* $\mathbb{R}^3$

The cancellation of the factor of ε^0 leads to problem $\mathscr{P}_0$ which reduces to

$$\frac{\partial\tau_n^0}{\partial r}=0 \quad \text{in } \Omega_0, \tag{44}$$

$$\tau_n^0=0 \quad \text{for } r=\pm 1, \tag{45}$$

$$\tau_t^0=0 \quad \text{for } s=s_- \text{ and } s=s_+. \tag{46}$$

Eq. (44) and boundary conditions (45) leads to

$$\tau_n^0=0$$

or equivalently in terms of mapping

$$\left(\overline{\frac{\partial\psi^2}{\partial r}}\frac{\partial\psi^1}{\partial s}\right)\frac{\partial\psi^1}{\partial s}+\left(\frac{\beta}{2}\left\|\frac{\partial\psi^1}{\partial s}\right\|^2+\frac{\beta+2}{2}\left\|\frac{\partial\psi^2}{\partial r}\right\|^2 +\frac{\beta}{2}\left\|\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}\right\|^2-\frac{3\beta+2}{2}\right)\frac{\partial\psi^2}{\partial r}+\left(\overline{\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}}\frac{\partial\psi^2}{\partial r}\right)\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}=0. \tag{47}$$

To conclude we must use the mass conservation law. According to the previous results, the expansion of Eq. (37) writes

$$\frac{\partial\psi^1}{\partial s}\cdot\left(\frac{\partial\psi^2}{\partial r}\wedge\frac{\partial\psi^0}{\partial x_3}\right)\geqslant a>0,$$

where $\wedge$ denotes the wedge product of $\mathbb{R}^3$. This condition implies that the vectors $\left(\partial\psi^1/\partial s, \partial\psi^2/\partial r, \mathrm{d}\psi^0/\mathrm{d}x_3\right)$ are linearly independent and constitute a basis of $\mathbb{R}^3$.

Therefore the projection of Eq. (47) on this basis leads to the following equations:

$$\overline{\frac{\partial\psi^2}{\partial r}}\frac{\partial\psi^1}{\partial s}=0, \tag{48}$$

$$\frac{\beta}{2}\left\|\frac{\partial\psi^1}{\partial s}\right\|^2+\frac{\beta+2}{2}\left\|\frac{\partial\psi^2}{\partial r}\right\|^2+\frac{\beta}{2}\left\|\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}\right\|^2-\frac{3\beta+2}{2}=0, \tag{49}$$

$$\overline{\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}}\frac{\partial\psi^2}{\partial r}=0 \tag{50}$$

which enable to write the vectors τ_t^0 and τ_3^0 on the following form:

$$\tau_t^0=\left\{2\frac{\beta+1}{\beta+2}\left\|\frac{\partial\psi^1}{\partial s}\right\|^2+\frac{\beta}{\beta+2}\left\|\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}\right\|^2 -\frac{(3\beta+2)}{(\beta+2)}\right\}\frac{\partial\psi^1}{\partial s}+\overline{\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}}\frac{\partial\psi^1}{\partial s}\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}, \tag{51}$$

$$\tau_3^0=\left\{\frac{\beta}{\beta+2}\left\|\frac{\partial\psi^1}{\partial s}\right\|^2+2\frac{\beta+1}{\beta+2}\left\|\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}\right\|^2 -\frac{(3\beta+2)}{(\beta+2)}\right\}\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}+\overline{\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}}\frac{\partial\psi^1}{\partial s}\frac{\partial\psi^1}{\partial s}. \tag{52}$$

The cancellation of the factor of ε^1 leads to problem $\mathscr{P}_1$ which writes

$$\frac{\partial\tau_n^1}{\partial r}+\frac{\partial\tau_t^0}{\partial s}=0 \quad \text{in } \Omega_0, \tag{53}$$

$$\tau_n^1=0 \quad \text{for } r=\pm 1, \tag{54}$$

$$\tau_t^1=0 \quad \text{for } s=s_- \text{ and } s=s_+. \tag{55}$$

Let us now integrate Eq. (53) upon the thickness. We have

$$\int_{-1}^{1}\frac{\partial\tau_t^0}{\partial s}\,\mathrm{d}r=0.$$

As τ_t^0 does not depend on r according to (51), using the boundary conditions (46), we get

$$\tau_t^0=0. \tag{56}$$

Then replacing τ_t^0 with its expression (51), as $(\partial\psi^1/\partial s, \partial\psi^2/\partial r, \partial\psi^0/\partial x_3)$ constitutes a basis of $\mathbb{R}^3$, we obtain

$$2\frac{\beta+1}{\beta+2}\left\|\frac{\partial\psi^1}{\partial s}\right\|^2+\frac{\beta}{\beta+2}\left\|\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}\right\|^2-\frac{(3\beta+2)}{(\beta+2)}=0, \tag{57}$$

$$\overline{\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}}\frac{\partial\psi^1}{\partial s}=0. \tag{58}$$

Eqs. (48), (50) and (58) implies that the vectors $\partial\psi^1/\partial s$, $\partial\psi^2/\partial r$ and $\mathrm{d}\psi^0/\mathrm{d}x_3$ constitute an orthogonal basis of $\mathbb{R}^3$.

Then using (57), expression (52) of τ_3^0 reduces to

$$\tau_3^0=\frac{3\beta+2}{2(\beta+1)}\left\{\left\|\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}\right\|^2-1\right\}\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}. \tag{59}$$

(v) *Let us prove that* $U^0=0$

As $\tau_t^0=0$ according to (56) Eq. (53) reduces to

$$\frac{\partial\tau_n^1}{\partial r}=0.$$

Using (54), we have in Ω_0:

$$\tau_n^1=0. \tag{60}$$

According to the previous results, problem $\mathscr{P}_2$ reduces to

$$\frac{\partial\tau_n^2}{\partial r}+\frac{\partial\tau_t^1}{\partial s}+\frac{\partial\tau_3^0}{\partial x_3}=0 \quad \text{in } \Omega_0, \tag{61}$$

$$\tau_n^2=0 \quad \text{for } r=\pm 1, \tag{62}$$

$$\tau_t^2=0 \quad \text{for } s=s_- \text{ and } s=s_+. \tag{63}$$

Let us integrate Eq. (61) with respect to r and s. Using (62) and (55), we obtain

$$\int_{s_-}^{s_+}\int_{-1}^{1}\frac{\mathrm{d}\tau_3^0}{\mathrm{d}x_3}\,\mathrm{d}r\,\mathrm{d}s=0.$$

Replacing τ_3^0 with its expression (59), we get

$$\frac{ES}{\mu}\frac{\mathrm{d}}{\mathrm{d}x_3}\left[\Delta_0\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}\right]=0, \tag{64}$$

where

$$\Delta_0=\left\|\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}\right\|^2-1$$

denotes the strain or the elongation of the middle line of the rod. To conclude we will write our problem on the form of a problem of minimization. Let us define the space $\mathscr{V}$ of "smooth" functions of $\mathbb{R}$ whose values are some vectors of $\mathbb{R}^3$. We then define the following space of mapping functions:

$$\mathscr{V}_0=\{\phi\in\mathscr{V}\text{ such as }\phi(0)=\phi(1)=0\}.$$

Then the weak formulation equivalent to Eq. (64) writes:

Find $\psi^0\in\mathscr{V}_0$ which satisfies to

$$\frac{ES}{\mu}\int_{-1}^{1}\Delta_0\frac{\mathrm{d}\psi}{\mathrm{d}x_3}\frac{\mathrm{d}v}{\mathrm{d}x_3}\,\mathrm{d}x_3=0,\quad\forall v\in\mathscr{V}_0. \tag{65}$$

It is easy to prove that the minimization problem associated to formulation (65) writes:

Find the mappings ψ which minimizes on $\mathscr{V}$ the functional:

$$J(\psi)=\frac{ES}{4\mu}\int_{-1}^{1}\Delta_0^2\,\mathrm{d}x_3.$$

It is trivial that the solution of this minimization problem satisfies to $\Delta_0=0$. This means that the length of the middle line of the rod does not vary with the mapping ψ^0. As the rod is assumed to be clamped at its extremities, we get

$$U^0=0.$$

This result proves that for the force level chosen, the reference scale for the displacements is not properly chosen. We must have $U_r=d_0$ in order the leading term of the expansion U^0 to be different from zero. However, we will not make a new dimensional analysis of equations with $U_r=d_0$, but we will keep on the asymptotic expansion of equations. And this, not to lose the reader.

On the other hand, before going on, let us define new parameters which enables to simplify the calculations and to interpret the results obtained. According to the previous results, we have

$$\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}=e_3$$

and (49) and (57) lead to

$$\left\|\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}\right\|^2=\left\|\frac{\partial\psi^1}{\partial s}\right\|^2=\left\|\frac{\partial\psi^2}{\partial r}\right\|^2=1. \tag{66}$$

This means that the vectors $\left(\partial\psi^1/\partial s,\partial\psi^2/\partial r,\mathrm{d}\psi^0/\mathrm{d}x_3\right)$ constitute an orthonormal basis of $\mathbb{R}^3$, and more precisely a basis of the deformed configuration at the leading order. Indeed the first non-zero term of the tangent mapping F writes

$$F^0=\left[\frac{\partial\psi^1}{\partial s}\quad\frac{\partial\psi^2}{\partial r}\quad\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}\right].$$

Thus the basis $(\partial\psi^1/\partial s,\partial\psi^2/\partial r,\mathrm{d}\psi^0/\mathrm{d}x_3)$ can be identified with Frenet basis of the deformed configuration at order zero. In this case, it is more convenient to set

$$\frac{\partial\psi^1}{\partial s}=T\quad\frac{\partial\psi^2}{\partial r}=N\quad\frac{\mathrm{d}\psi^0}{\mathrm{d}x_3}=e_3,$$

where T and N represent the tangent and normal vector to the curve $\mathscr{C}$ at the point $m=\psi^1(m_0)$ in the deformed configuration. In what follows (T,N,e_3) will denote the local basis in the deformed configuration.

On the other hand, as N does not depend on r, we set

$$\psi^2=U^2(s,x_3)+rN. \tag{67}$$

In the same way, we have

$$\psi^1=u_3^1(x_3)e_3+X(s,x_3), \tag{68}$$

where X denotes the position vector at current point $m=\psi^1(m_0)$ in the plane of a section. It satisfies to

$$X.e_3=0,$$

$$\frac{\partial X}{\partial s}=T.$$

Moreover, we have

$$\frac{\partial T}{\partial s}=cN,$$

$$\frac{\partial N}{\partial s}=-cT,$$

where c denotes the curvature of the profile $\mathscr{C}$ in the deformed configuration.[5] □

Lemma 2. *For levels of applied forces such as $\mathscr{F}_t=\mathscr{F}_n=\varepsilon^5$, $\mathscr{G}_t=\mathscr{G}_n=\varepsilon^5$ and $\mathscr{F}_3=\mathscr{G}_3=\varepsilon^4$, the term of order* 1 *of the expansion of the axial displacement is equal to zero*:

$$u_3^1=0.$$

Proof. First, condition (66) and expression (59) of τ_3^0 lead to

$$\tau_3^0=0.$$

[5] According to relations (66), we have $\|\partial\psi^1/\partial s\|=1$. There is no variation of the metric of the profile at the leading order. So it is possible to keep the curvilinear abscisse s to parameter the profile $\mathscr{C}$ in the deformed configuration.

On the other hand, according to (60) we have $\tau_n^1 = 0$. Equivalently in terms of mapping, using the fact that (T, N, e_3) is a basis of the deformed configuration, we get

$$\overline{\frac{\partial \psi^3}{\partial r}} T + \overline{\frac{\partial \psi^2}{\partial s}} N = 0, \tag{69}$$

$$\beta \left[r c_0 + \overline{\frac{\partial \psi^2}{\partial s}} T \right] + (\beta + 2) \overline{\frac{\partial \psi^3}{\partial r}} N + \beta \overline{\frac{\partial \psi^1}{\partial x_3}} e_3 = 0, \tag{70}$$

$$\overline{\frac{\partial \psi^1}{\partial x_3}} N + \overline{\frac{\partial \psi^3}{\partial r}} e_3 = 0. \tag{71}$$

Then we obtain the following expressions of the stress vectors:

$$\tau_t^1 = \left(4 \frac{\beta+1}{\beta+2} \left[\overline{\frac{\partial \psi^2}{\partial s}} T + r c_0 \right] + \frac{2\beta}{\beta+2} \overline{\frac{\partial \psi^1}{\partial x_3}} e_3 \right) T + \left(\overline{\frac{\partial \psi^1}{\partial x_3}} T + \overline{\frac{\partial \psi^2}{\partial s}} e_3 \right) e_3,$$

$$\tau_3^1 = \left(\frac{2\beta}{\beta+2} \left[\overline{\frac{\partial \psi^2}{\partial s}} T + r c_0 \right] + 4 \frac{\beta+1}{\beta+2} \overline{\frac{\partial \psi^1}{\partial x_3}} e_3 \right) e_3 + \left(\overline{\frac{\partial \psi^1}{\partial x_3}} T + \overline{\frac{\partial \psi^2}{\partial s}} e_3 \right) T.$$

Now replacing ψ^2 with its expression (67), we get

$$\tau_t^1 = \left(4 \frac{\beta+1}{\beta+2} \overline{\frac{\partial U^2}{\partial s}} T + 2 \frac{\beta}{\beta+2} \overline{\frac{\partial \psi^1}{\partial x_3}} e_3 \right) T + \left(\overline{\frac{\partial \psi^1}{\partial x_3}} T + \overline{\frac{\partial U^2}{\partial s}} e_3 \right) e_3 + 4 \frac{\beta+1}{\beta+2} r (c_0 - c) T, \tag{72}$$

$$\tau_3^1 = \left(2 \frac{\beta}{\beta+2} \overline{\frac{\partial U^2}{\partial s}} T + 4 \frac{\beta+1}{\beta+2} \overline{\frac{\partial \psi^1}{\partial x_3}} e_3 \right) e_3 + \left(\overline{\frac{\partial \psi^1}{\partial x_3}} T + \overline{\frac{\partial U^2}{\partial s}} e_3 \right) T + 2 \frac{\beta}{\beta+2} r (c_0 - c) e_3. \tag{73}$$

The new problem $\mathscr{P}_2$

According to the previous results, problem $\mathscr{P}_2$ now reduces to

$$\frac{\partial \tau_n^2}{\partial r} + \frac{\partial \tau_t^1}{\partial s} = 0 \quad \text{in } \Omega_0. \tag{74}$$

Using boundary conditions (62), let us integrate Eq. (74) upon the thickness. We obtain

$$\int_{-1}^{1} \frac{\partial \tau_t^1}{\partial s} \, dr = 0$$

which reduces in Ω_0 to

$$\int_{-1}^{1} \tau_t^1 \, dr = 0$$

according to the boundary conditions (55). Then replacing τ_t^1 with its expression (72), we get

$$4 \frac{\beta+1}{\beta+2} \overline{\frac{\partial U^2}{\partial s}} T + 2 \frac{\beta}{\beta+2} \overline{\frac{\partial \psi^1}{\partial x_3}} e_3 = 0, \tag{75}$$

$$\overline{\frac{\partial \psi^1}{\partial x_3}} T + \overline{\frac{\partial U^2}{\partial s}} e_3 = 0. \tag{76}$$

Using these new equalities, expressions (72) and (73) of τ_t^1 and τ_3^1 reduce to

$$\tau_t^1 = 4 \frac{\beta+1}{\beta+2} r (c_0 - c) T, \tag{77}$$

$$\tau_3^1 = \tau_{33}^1 e_3 = \left(\frac{E}{\mu} \frac{du_3^1}{dx_3} + 2 \frac{\beta}{\beta+2} r (c_0 - c) \right) e_3. \tag{78}$$

Finally, using the last expression of τ_t^1, the boundary condition (62), and Eq. (74), we obtain the following expression of τ_n^2:

$$\tau_n^2 = -4 \frac{\beta+1}{\beta+2} \frac{r^2-1}{2} \left(\frac{\partial (c_0 - c)}{\partial s} T + c (c - c_0) N \right). \tag{79}$$

On the other hand, let us multiply Eq. (74) with r and integrate it upon the thickness. 0

$$\int_{-1}^{1} \left(r \frac{\partial \tau_n^2}{\partial r} + r \frac{\partial \tau_t^1}{\partial s} \right) dr = 0.$$

Using (62), an integration by parts of the first term leads to

$$\int_{-1}^{1} \left(-\tau_n^2 + r \frac{\partial \tau_t^1}{\partial s} \right) dr = 0. \tag{80}$$

This equality will be used later.

Problem $\mathscr{P}_3$

The cancellation of the factor of ε^3 leads to problem $\mathscr{P}_3$:

$$\frac{\partial \tau_n^3}{\partial r} + \frac{\partial \tau_t^2}{\partial s} + r c_0 \frac{\partial \tau_t^1}{\partial s} - c_0 \tau_n^2 + \frac{\partial \tau_3^1}{\partial x_3} = 0 \quad \text{in } \Omega_0, \tag{81}$$

$$\tau_n^3 = 0 \quad \text{for } r = \pm 1, \tag{82}$$

$$\tau_t^3 = 0 \quad \text{for } s = s_- \text{ and } s = s_+. \tag{83}$$

Using (82) and (63), an integration of Eq. (81) with respect to r and s leads to

$$\int_{s_-}^{s_+} \int_{-1}^{1} \left(r c_0 \frac{\partial \tau_t^1}{\partial s} - c_0 \tau_n^2 + \frac{\partial \tau_3^1}{\partial x_3} \right) dr \, ds = 0$$

which reduces to

$$\int_{s_-}^{s_+} \int_{-1}^{1} \frac{\partial \tau_3^1}{\partial x_3} \, dr \, ds = 0$$

according to (80). Then replacing τ_3^1 with its expression (78), we get

$$\frac{ES}{\mu} \frac{d^2 u_3^1}{dx_3^2} = 0.$$

According to the boundary conditions (the rod is assumed to be clamped at its extremities), we obtain

$$u_3^1 = 0$$

which concludes the proof of Lemma 2. Therefore, for the force level considered, the reference scale for the axial displacement is not properly chosen. We have to consider $u_{3r} = h_0$ in order the leading term u_3^0 of the expansion of U to be different from zero. Once again, to simplify the notations and the following calculations, we will not make a new dimensional analysis of the equations. □

5. A non-linear model of thin-walled rod for strongly bent profile

5.1. Non-linear displacement of Vlassov type

Result 1. *For force levels such as $\mathscr{F}_t = \mathscr{F}_n = \varepsilon^5$, $\mathscr{G}_t = \mathscr{G}_n = \varepsilon^5$ et $\mathscr{F}_3 = \mathscr{G}_3 = \varepsilon^4$, the first non-zero term of the expansion of the displacement is $U = (U^1, u_3^2)$. The associated kinematics is non-linear of Vlassov type and verify*

$$\left| \begin{array}{l} U^1 = V(x_3) + (R_\Theta - I_2)x, \\ u_3^2 = u_3(x_3) - \overline{R_\Theta x}\dfrac{\mathrm{d}V}{\mathrm{d}x_3} - \omega_n \dfrac{\mathrm{d}\Theta}{\mathrm{d}x_3}, \end{array} \right.$$

where $V(x_3)$ and $u_3(x_3)$ denote the bending and the axial displacement of the point G, respectively, Θ the angle of the rotation around the (G, e_3) axis whose matrix of rotation is $R_\Theta = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix}$, ω_n the sectorial surface given by $\mathrm{d}\omega_n/\mathrm{d}s = -q$ *and I_2 the identity or $\mathbb{R}^2$.*

Proof. In a first time, we prove that the sections of the rod behave like rigid solids (the curvature and the metric do not vary during the deformation). We then deduce the expression of the displacement.

(i) *The sections stay plane and dot not deform*

We just proved that $u_3^1 = 0$. According to (68) and (75), we have

$$\frac{\partial \bar{U}^2}{\partial s} T = 0. \tag{84}$$

Let us integrate Eq. (81) upon the thickness. Using (82) and (80), we obtain

$$\int_{-1}^{1} \left(\frac{\partial \tau_t^2}{\partial s} + \frac{\partial \tau_3^1}{\partial x_3} \right) \mathrm{d}r = 0$$

whose projection onto (T, N) basis leads to

$$\int_{-1}^{1} \left(\frac{\partial \tau_{tT}^2}{\partial s} - c\tau_{tN}^2 \right) \mathrm{d}r = 0, \tag{85}$$

$$\int_{-1}^{1} \left(\frac{\partial \tau_{tN}^2}{\partial s} + c\tau_{tT}^2 \right) \mathrm{d}r = 0. \tag{86}$$

Let us now derive Eq. (86) with respect to s. We get

$$\int_{-1}^{1} \left(\frac{\partial^2 \tau_{tN}^2}{\partial s^2} + \frac{\partial c}{\partial s}\tau_{tT}^2 + c\frac{\partial \tau_{tT}^2}{\partial s} \right) \mathrm{d}r = 0$$

which becomes using (85) and (86)

$$\int_{-1}^{1} \left(\frac{\partial^2 \tau_{tN}^2}{\partial s^2} - \frac{1}{c}\frac{\partial c}{\partial s}\frac{\partial \tau_{tN}^2}{\partial s} + c^2\tau_{tN}^2 \right) \mathrm{d}r = 0. \tag{87}$$

To obtain this equation, we assumed that the curvature of the deformed configuration was different from zero: $c \neq 0$. This assumption is satisfied because the curvature must stay of the same order of magnitude along the axis of the rod.[6] Indeed, the rod being clamped at its extremities, we have at this points $c = c_0$ which is large (we consider here only thin-walled rods with strongly curved profile). Thus with the one-scale approach developed here, the curvature cannot get the value zero in the case of thin-walled rods with strongly curved profile.

Let us now use the following relation:

$$\int_{-1}^{1} \tau_{tN}^2 \,\mathrm{d}r = \int_{-1}^{1} \tau_{nT}^2 \,\mathrm{d}r \tag{88}$$

necessary for the calculations whose proof is detailed in Appendix A. Replacing τ_{tN}^2 with τ_{nT}^2 in Eq. (87), we get

$$\int_{-1}^{1} \left(\frac{\partial^2 \tau_{nT}^2}{\partial s^2} - \frac{1}{c}\frac{\partial c}{\partial s}\frac{\partial \tau_{nT}^2}{\partial s} + c^2\tau_{nT}^2 \right) \mathrm{d}r = 0.$$

Finally replacing τ_{nT}^2 with its expression (79), we obtain the following second-order differential equation

$$\frac{\partial^3(c_0 - c)}{\partial s^3} - \frac{1}{c}\frac{\partial c}{\partial s}\frac{\partial^2(c_0 - c)}{\partial s^2} + c^2\frac{\partial(c_0 - c)}{\partial s} = 0$$

whose general solution writes

$$\frac{\partial(c_0 - c)}{\partial s} = A(x_3)\cos(\alpha) + B(x_3)\sin(\alpha) \tag{89}$$

with $\mathrm{d}\alpha/\mathrm{d}s = c$.

The boundary condition (55) and expression (77) of τ_t^1 give us

$$c_0 - c = 0 \quad \text{for } s = s_- \text{ and } s = s_+.$$

In the same way, (63), (127) and (79) lead to

$$\frac{\partial(c_0 - c)}{\partial s} = 0 \quad \text{for } s = s_- \text{ and } s = s_+.$$

With the previous boundary conditions, the unique solution of (89) writes

$$c = c_0.$$

Thus we proved that the curvature does not vary during the deformation. According to (66) the metric of the profile does not

[6] If not we have multi-scale problems which are much more complicated and not studied in this paper.

vary neither at the first order. The displacement of the section is a plane rigid solid displacement. This assumption made by Vlassov in linear theory stays valid in non-linear theory, for the level of displacements considered here.

Therefore the local basis (T, N, e_3) of the deformed configuration can be deduced from the initial local basis (t, n, e_3) by a rotation around the e_3 axis which does not depend on s.

Let us denote $\Theta(x_3)$ the angle of this rotation whose two-dimensional matrix writes in the (t, n) basis:

$$R_\Theta = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix}$$

then we have

$$\frac{\partial \psi^1}{\partial s} = T = R_\Theta t$$

and as $\psi^1 = U^1 + x$, we deduce that

$$\frac{\partial U^1}{\partial s} = (R_\Theta - I_2)t,$$

where I_2 denotes the identity matrix of the plane (t, n). So that the displacement of the section in its plane writes

$$U^1 = V(x_3) + (R_\Theta - I_2)x, \tag{90}$$

where $V(x_3)$ denotes the displacement of the origin of the frame. In the system of principal coordinates, the origin considered coincides with the center of gravity G of the section.

Let us notice that it is possible to introduce equally the shear center C of the section. To do this, let us writes

$$U^1 = W(x_3) + (R_\Theta - I_2)(x - x_c),$$

where x_c denotes the position of the point C in the plane of the section. The displacement $W(x_3)$ then denotes the displacement of the shear center C. In what follows, to simplify the notations, we will use the expression (90) of U^1.

(ii) *Expression of the axial displacement*

To determine the axial displacement u_3^2, let us use (76) and (67). We obtain

$$\overline{\frac{\partial \psi^1}{\partial x_3}} T + \overline{\frac{\partial \psi^2}{\partial s}} e_3 = \overline{\frac{\partial \psi^1}{\partial x_3}} T + \frac{\partial u_3^2}{\partial s} = 0.$$

As $T = \mathrm{d}\psi^1/\mathrm{d}s$, we deduce that

$$\frac{\partial u_3^2}{\partial s} = -\overline{\frac{\partial \psi^1}{\partial x_3}}\,\frac{\partial \psi^1}{\partial s}$$

or equivalently

$$\frac{\partial u_3^2}{\partial s} = -\frac{\partial}{\partial s}\left[\overline{\frac{\partial \psi^1}{\partial x_3}}\psi^1\right] + \overline{\frac{\partial T}{\partial x_3}}\psi^1.$$

As $\partial T/\partial x_3 = (\mathrm{d}\Theta/\mathrm{d}x_3)N$, we obtain

$$\frac{\partial u_3^2}{\partial s} = -\frac{\partial}{\partial s}\left[\overline{\frac{\partial \psi^1}{\partial x_3}}\psi^1\right] + Q\frac{\mathrm{d}\Theta}{\mathrm{d}x_3},$$

where the vector $\Pi\psi^1$ (projection of ψ^1 onto the (t, n) basis) has been decomposed onto the local basis (T, N) as follows: $\Pi\psi^1 = HT + QN$. Then we deduce that

$$u_3^2 = \bar{u}_3^2 - \overline{\frac{\partial \psi^1}{\partial x_3}}\psi^1 - \omega_N\frac{\mathrm{d}\Theta}{\mathrm{d}x_3},$$

where $\partial\omega_N/\partial s = -Q$, and $\bar{u}_3^2$ denotes the axial displacement of the section. Let us notice that ω_N corresponds to the classical sectorial area (see [2]) but calculated here in the deformed configuration.

It is possible to express the displacement u_3^2 in a more classical way. To do this, let us develop the previous expression. Replacing ψ^1 by its expression (90), we obtain

$$u_3^2 = \bar{u}_3^2 - \left[\overline{V} + \overline{R_\Theta x}\right]\left[\frac{\mathrm{d}V}{\mathrm{d}x_3} + \frac{\mathrm{d}\Theta}{\mathrm{d}x_3}\Lambda R_\Theta x\right] - \omega_N\frac{\mathrm{d}\Theta}{\mathrm{d}x_3},$$

where Λ denotes the two-dimensional matrix of "the wedge product with e_3", which writes in all basis orthogonal to e_3 (in particular in (t, n)):

$$\Lambda = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Let us develop the previous expression of u_3^2. As $\overline{R_\Theta x}\Lambda R_\Theta x = 0$, we obtain

$$u_3^2 = \bar{u}_3^2 - \overline{V}\frac{\mathrm{d}V}{\mathrm{d}x_3} - \overline{R_\Theta x}\frac{\mathrm{d}V}{\mathrm{d}x_3} - \frac{\mathrm{d}\Theta}{\mathrm{d}x_3}\overline{V}\Lambda R_\Theta x - \omega_N\frac{\mathrm{d}\Theta}{\mathrm{d}x_3}$$

or equivalently

$$u_3^2 = u_3(x_3) - \bar{x}\overline{R}_\Theta\frac{\mathrm{d}V}{\mathrm{d}x_3} - \omega_n\frac{\mathrm{d}\Theta}{\mathrm{d}x_3},$$

where we set

$$u_3(x_3) = \bar{u}_3^2 - \overline{V}\frac{\mathrm{d}V}{\mathrm{d}x_3},$$

$$\omega_n = \overline{V}\Lambda R_\Theta x + \omega_N.$$

Let us notice that ω_n denotes the sectorial area of the non-deformed configuration. Indeed, we have

$$\frac{\mathrm{d}\omega_n}{\mathrm{d}s} = \overline{V}\Lambda R_\Theta t - \frac{\mathrm{d}\omega_N}{\mathrm{d}s}.$$

As $\mathrm{d}\omega_N/\mathrm{d}s = -Q$, $Q = \overline{\psi^1}N$ and $\Lambda t = n$, we get

$$\frac{\mathrm{d}\omega_n}{\mathrm{d}s} = \overline{V}R_\Theta n - \overline{\psi^1}N.$$

On the other hand, according to (90), we have $\psi^1 = V + R_\Theta x$ and $R_\Theta n = N$. So that

$$\frac{\mathrm{d}\omega_n}{\mathrm{d}s} = -\bar{x}n = -q$$

which corresponds to the classical definition of the sectorial area in the non-deformed configuration [2]. □

5.2. Computation of the stresses

Result 2. *For external level forces such as* $\mathscr{F}_t = \mathscr{F}_n = \varepsilon^5$, $\mathscr{G}_t = \mathscr{G}_n = \varepsilon^5$ *and* $\mathscr{F}_3 = \mathscr{G}_3 = \varepsilon^4$, *the first non-zero term of the stresses* σ *is* σ^2 *whose expression in* (t, n, e_3) *basis writes*

$$\sigma^2 = \begin{bmatrix} 0 & 0 & \sigma^2_{t3} \\ 0 & 0 & 0 \\ \sigma^2_{t3} & 0 & \sigma^2_{33} \end{bmatrix}$$

with

$$\sigma^2_{t3} = -2r\frac{\mathrm{d}\Theta}{\mathrm{d}x_3}$$

$$\sigma^2_{33} = \frac{E}{\mu}\frac{\partial u^2_3}{\partial x_3} + \frac{E}{2\mu}\left\|\frac{\partial \psi^1}{\partial x_3}\right\|^2.$$

Proof. Let us first explain ψ^3 and then the components τ^2 and σ^2 of the stresses at order two.

Expression of ψ^3

As we proved that $c = c_0$, we have according to (77) and (79):

$$\tau^1_t = \tau^2_n = 0.$$

On the other hand, from Eqs. (69), (70) and (71), using (84) and the result of Lemma 2, we deduce that ψ^3 satisfy the three following relations:

$$\overline{\frac{\partial \psi^3}{\partial r}}T = -\overline{\frac{\partial \psi^2}{\partial s}}N = -\overline{\frac{\partial U^2}{\partial s}}N,$$

$$\overline{\frac{\partial \psi^3}{\partial r}}N = 0,$$

$$\overline{\frac{\partial \psi^3}{\partial r}}e_3 = -\overline{\frac{\partial \psi^1}{\partial x_3}}N.$$

Thus the components of ψ^3 are given by

$$\psi^3_T = \tilde{U}^3_T - r\overline{\frac{\partial U^2}{\partial s}}N, \tag{91}$$

$$\psi^3_N = \tilde{U}^3_N, \tag{92}$$

$$\psi^3_3 = \tilde{U}^3_3 - r\overline{\frac{\partial \psi^1}{\partial x_3}}N. \tag{93}$$

On the other hand, as $\tau^2_n = 0$, we have the following relations:

$$\overline{\frac{\partial \psi^4}{\partial r}}T + \overline{\frac{\partial \psi^3}{\partial r}}\frac{\partial \psi^2}{\partial s} + \overline{\frac{\partial \psi^3}{\partial s}}N = 0,$$

$$\overline{\frac{\partial \psi^4}{\partial r}}e_3 + \overline{\frac{\partial \psi^3}{\partial r}}\frac{\partial \psi^1}{\partial x_3} + \overline{\frac{\partial \psi^2}{\partial x_3}}N = 0,$$

$$\frac{\beta}{2}\left[\left\|\frac{\partial \psi^2}{\partial s}\right\|^2 + 2\overline{\frac{\partial \psi^3}{\partial s}}T + 4rc_0\overline{\frac{\partial \psi^2}{\partial s}}T + 3(rc_0)^2\right] + \frac{\beta+2}{2}\left[\left\|\frac{\partial \psi^3}{\partial r}\right\|^2 + 2\overline{\frac{\partial \psi^4}{\partial r}}N\right] + \frac{\beta}{2}\left[\left\|\frac{\partial \psi^1}{\partial x_3}\right\|^2 + 2\overline{\frac{\partial \psi^2}{\partial x_3}}e_3\right] = 0$$

which leads to

$$\tau^2_t = \left(2\frac{\beta+1}{\beta+2}\left[\left\|\frac{\partial \psi^2}{\partial s}\right\|^2 + 2\overline{\frac{\partial \psi^3}{\partial s}}T + 4rc_0\overline{\frac{\partial \psi^2}{\partial s}}T + 3(rc_0)^2\right] + \frac{\beta}{\beta+2}\left[\left\|\frac{\partial \psi^1}{\partial x_3}\right\|^2 + 2\overline{\frac{\partial \psi^2}{\partial x_3}}e_3\right]\right)T + \left(\overline{\frac{\partial \psi^2}{\partial x_3}}T + \overline{\frac{\partial \psi^1}{\partial x_3}}\frac{\partial \psi^2}{\partial s} + \overline{\frac{\partial \psi^3}{\partial s}}e_3\right)e_3,$$

$$\tau^2_3 = \left(\frac{\beta}{\beta+2}\left[\left\|\frac{\partial \psi^2}{\partial s}\right\|^2 + 2\overline{\frac{\partial \psi^3}{\partial s}}T + 4rc_0\overline{\frac{\partial \psi^2}{\partial s}}T + 3(rc_0)^2\right] + 2\frac{\beta+1}{\beta+2}\left[\left\|\frac{\partial \psi^1}{\partial x_3}\right\|^2 + 2\overline{\frac{\partial \psi^2}{\partial x_3}}e_3\right]\right)e_3 + \left(\overline{\frac{\partial \psi^2}{\partial x_3}}T + \overline{\frac{\partial \psi^1}{\partial x_3}}\frac{\partial \psi^2}{\partial s} + \overline{\frac{\partial \psi^3}{\partial s}}e_3\right)T.$$

Then replacing ψ^2 and ψ^3 with their expressions (67) and (91)–(93), respectively, we obtain

$$\tau^2_t = -4r\frac{\beta+1}{\beta+2}\frac{\partial}{\partial s}\left(\overline{\frac{\partial U^2}{\partial s}}N\right)T + 2\frac{\beta+1}{\beta+2} \times \left[\left\|\frac{\partial U^2}{\partial s}\right\|^2 + 2\overline{\frac{\partial \tilde{U}^3_T}{\partial s}} - 2c_0\tilde{U}^3_N\right]T + \frac{\beta}{\beta+2}\left[\left\|\frac{\partial \psi^1}{\partial x_3}\right\|^2 + 2\overline{\frac{\partial U^2}{\partial x_3}}e_3\right]T + \left(\overline{\frac{\partial U^2}{\partial x_3}}T + \overline{\frac{\partial \psi^1}{\partial x_3}}\frac{\partial U^2}{\partial s} + \overline{\frac{\partial \tilde{U}^3}{\partial s}}e_3 - 2r\frac{\mathrm{d}\Theta}{\mathrm{d}x_3}\right)e_3, \tag{94}$$

$$\tau^2_3 = -2r\frac{\beta}{\beta+2}\frac{\partial}{\partial s}\left(\overline{\frac{\partial U^2}{\partial s}}N\right)e_3 + \frac{\beta}{\beta+2} \times \left[\left\|\frac{\partial U^2}{\partial s}\right\|^2 + 2\overline{\frac{\partial \tilde{U}^3_T}{\partial s}} - 2c_0\tilde{U}^3_N\right]e_3 + 2\frac{\beta+1}{\beta+2}\left[\left\|\frac{\partial \psi^1}{\partial x_3}\right\|^2 + 2\overline{\frac{\partial U^2}{\partial x_3}}e_3\right]e_3 + \left(\overline{\frac{\partial U^2}{\partial x_3}}T + \overline{\frac{\partial \psi^1}{\partial x_3}}\frac{\partial U^2}{\partial s} + \overline{\frac{\partial \tilde{U}^3}{\partial s}}e_3 - 2r\frac{\mathrm{d}\Theta}{\mathrm{d}x_3}\right)T, \tag{95}$$

The new problem $\mathscr{P}_3$

According to the previous results, problem $\mathscr{P}_3$ reduces to

$$\frac{\partial\tau_n^3}{\partial r}+\frac{\partial\tau_t^2}{\partial s}=0 \quad \text{in } \Omega_0. \tag{96}$$

Using (82), an integration of Eq. (96) upon the thickness leads to

$$\int_{-1}^{1}\frac{\partial\tau_t^2}{\partial s}\,\mathrm{d}r=0$$

which reduces to

$$\int_{-1}^{1}\tau_t^2\,\mathrm{d}r=0$$

according to (63). Replacing τ_t^2 with its expression (94), we obtain

$$2\frac{\beta+1}{\beta+2}\left[\left\|\frac{\partial U^2}{\partial s}\right\|^2+2\overline{\frac{\partial\tilde{U}_T^3}{\partial s}}-2c_0\tilde{U}_N^3\right]+\frac{\beta}{\beta+2}\left[\left\|\frac{\partial\psi^1}{\partial x_3}\right\|^2+2\overline{\frac{\partial U^2}{\partial x_3}e_3}\right]=0,$$

$$\overline{\overline{\frac{\partial U^2}{\partial x_3}}T+\overline{\frac{\partial\psi^1}{\partial x_3}}\frac{\partial U^2}{\partial s}-\overline{\frac{\partial\tilde{U}^3}{\partial s}}e_3}=0.$$

Thus the expressions of τ_t^2 and τ_3^2 reduce to

$$\tau_t^2=-4r\frac{\beta+1}{\beta+2}\frac{\partial}{\partial s}\left(\overline{\frac{\partial U^2}{\partial s}}N\right)T-2r\frac{\mathrm{d}\Theta}{\mathrm{d}x_3}e_3, \tag{97}$$

$$\tau_3^2=-2r\frac{\beta}{\beta+2}\frac{\partial}{\partial s}\left(\overline{\frac{\partial U^2}{\partial s}}N\right)e_3+\frac{E}{2\mu}\left[\left\|\frac{\partial\psi^1}{\partial x_3}\right\|^2+2\overline{\frac{\partial U^2}{\partial x_3}e_3}\right]e_3-2r\frac{\mathrm{d}\Theta}{\mathrm{d}x_3}T. \tag{98}$$

Finally, using (96) and (98), we deduce the following expression of τ_n^3:

$$\tau_n^3=4\frac{\beta+1}{\beta+2}\left[\frac{\partial^2}{\partial s^2}\left(\overline{\frac{\partial U^2}{\partial s}}N\right)T+c_0\frac{\partial}{\partial s}\left(\overline{\frac{\partial U^2}{\partial s}}N\right)N\right]\frac{r^2-1}{2}. \tag{99}$$

On the other hand, in the same way as for problem $\mathscr{P}_2$, let us multiply Eq. (96) with r and integrate it upon the thickness. We obtain

$$\int_{-1}^{1}\left(r\frac{\partial\tau_n^3}{\partial r}+r\frac{\partial\tau_t^2}{\partial s}\right)\mathrm{d}r=0.$$

Using (82), an integration by parts of the first term leads to

$$\int_{-1}^{1}\left(-\tau_n^3+r\frac{\partial\tau_t^2}{\partial s}\right)\mathrm{d}r=0 \tag{100}$$

which will be used later.

The problem $\mathscr{P}_4$

The cancellation of the factor of ε^4 leads to problem $\mathscr{P}_4$ which writes

$$\frac{\partial\tau_n^4}{\partial r}+\frac{\partial\tau_t^3}{\partial s}+rc_0\frac{\partial\tau_t^2}{\partial s}-c_0\tau_n^3+\frac{\partial\tau_3^2}{\partial x_3}=-f_3e_3 \quad \text{in } \Omega_0, \tag{101}$$

$$\tau_n^4=g_3^{\pm}e_3 \quad \text{for } r=\pm1, \tag{102}$$

$$\tau_t^4=0 \quad \text{for } s=s_- \text{ and } s=s_+. \tag{103}$$

Let us now project Eq. (101) onto the plane of a section and integrate it upon the thickness. Using (100) and (102), we get

$$\int_{-1}^{1}\Pi\left(\frac{\partial\tau_t^3}{\partial s}+\frac{\partial\tau_3^2}{\partial x_3}\right)\mathrm{d}r=0,$$

where we recall that Π denotes the orthogonal projection onto the plane of a section. Decomposing this equation in the basis (T,N), we obtain the two following scalar equations:

$$\int_{-1}^{1}\left(\frac{\partial\tau_{tT}^3}{\partial s}-c_0\tau_{tN}^3\right)\mathrm{d}r=0, \tag{104}$$

$$\int_{-1}^{1}\left(\frac{\partial\tau_{tN}^3}{\partial s}+c_0\tau_{tT}^3\right)\mathrm{d}r=0. \tag{105}$$

Let us derive Eq. (105) with respect to s. We have

$$\int_{-1}^{1}\left(\frac{\partial^2\tau_{tN}^3}{\partial s^2}+\frac{\partial c_0}{\partial s}\tau_{tT}^3+c_0\frac{\partial\tau_{tT}^3}{\partial s}\right)\mathrm{d}r=0$$

which can be written

$$\int_{-1}^{1}\left(\frac{\partial^2\tau_{tN}^3}{\partial s^2}-\frac{1}{c_0}\frac{\partial c_0}{\partial s}\frac{\partial\tau_{tN}^3}{\partial s}+c_0^2\tau_{tN}^3\right)\mathrm{d}r=0 \tag{106}$$

using (104) and (105). To simplify this last equation, let us use the following relation:

$$\int_{-1}^{1}\tau_{tN}^3\,\mathrm{d}r=\int_{-1}^{1}\tau_{nT}^3\,\mathrm{d}r. \tag{107}$$

whose proof is detailed in Appendix A.

Using the previous relation, Eq. (106) becomes

$$\int_{-1}^{1}\left(\frac{\partial^2\tau_{nT}^3}{\partial s^2}-\frac{1}{c_0}\frac{\partial c_0}{\partial s}\frac{\partial\tau_{nT}^3}{\partial s}+c_0^2\tau_{nT}^3\right)\mathrm{d}r=0.$$

Now replacing τ_{nT}^3 with its expression (99), we obtain

$$\frac{\partial^4}{\partial s^4}\left(\overline{\frac{\partial U^2}{\partial s}}N\right)-\frac{1}{c_0}\frac{\partial c_0}{\partial s}\frac{\partial^3}{\partial s^3}\left(\overline{\frac{\partial U^2}{\partial s}}N\right)+c_0^2\frac{\partial^2}{\partial s^2}\left(\overline{\frac{\partial U^2}{\partial s}}N\right)=0$$

whose general solution writes

$$\frac{\partial^2}{\partial s^2}\left(\overline{\frac{\partial U^2}{\partial s}}N\right)=A(x_3)\cos(\alpha)+B(x_3)\sin(\alpha)$$

with

$$\frac{\mathrm{d}\alpha}{\mathrm{d}s} = c_0.$$

On the other hand, the boundary condition (63) and expression (97) of τ_t^2 lead to

$$\tau_{tT}^2 = 0 \quad \text{for } s = s_- \text{ and } s = s_+$$

or equivalently

$$\frac{\partial}{\partial s}\left(\overline{\frac{\partial U^2}{\partial s}N}\right) = 0 \quad \text{for } s = s_- \text{ and } s = s_+.$$

In the same way, (83) implies that

$$\tau_{tN}^3 = 0 \quad \text{for } s = s_- \text{ and } s = s_+$$

which leads, according to relation (2), to

$$\int_{-1}^{1} \tau_{nT}^3 \, \mathrm{d}r = 0.$$

Then replacing τ_{nT}^3 with its expression (99), we get

$$\frac{\partial^2}{\partial s^2}\left(\overline{\frac{\partial U^2}{\partial s}N}\right) = 0 \quad \text{for } s = s_- \text{ and } s = s_+$$

which leads to

$$\frac{\partial}{\partial s}\left(\overline{\frac{\partial U^2}{\partial s}N}\right) = 0$$

according to the four boundary condition above. Thus we have

$$\overline{\frac{\partial U^2}{\partial s}N} = \Psi(x_3). \tag{108}$$

Finally, we have $\tau_n^3 = 0$ and the stresses τ_t^2 and τ_3^2 reduce to

$$\tau_t^2 = \sigma_{t3}^2 e_3 = -2r\frac{\mathrm{d}\Theta}{\mathrm{d}x_3}e_3, \tag{109}$$

$$\begin{aligned}\tau_3^2 &= \sigma_{t3}^2 T + \sigma_{33}^2 e_3 \\ &= -2r\frac{\mathrm{d}\Theta}{\mathrm{d}x_3}T + \frac{E}{2\mu}\left[\left\|\frac{\partial\psi^1}{\partial x_3}\right\|^2 + 2\overline{\frac{\partial U^2}{\partial x_3}e_3}\right]e_3\end{aligned} \tag{110}$$

which leads to the expressions of Result 2. □

5.3. One-dimensional equilibrium equations

Result 3 (*Traction equation*). *For a level of applied forces such as $\mathscr{F}_t = \mathscr{F}_n = \varepsilon^5$, $\mathscr{G}_t = \mathscr{G}_n = \varepsilon^5$ and $\mathscr{F}_3 = \mathscr{G}_3 = \varepsilon^4$, the dimensionless unknowns of the displacement field $V(x_3)$, $u_3(x_3)$ and $\Theta(x_3)$ are solution of the following traction equation*:

$$\frac{E}{\mu}\frac{\mathrm{d}}{\mathrm{d}x_3}\left[S\frac{\mathrm{d}u_3}{\mathrm{d}x_3} - \overline{R_\Theta S_x}\frac{\mathrm{d}^2V}{\mathrm{d}x_3^2} - S_{\omega_n}\frac{\mathrm{d}^2\Theta}{\mathrm{d}x_3^2} + \frac{1}{2}\left(S\left\|\frac{\mathrm{d}V}{\mathrm{d}x_3}\right\|^2 + I_x\left(\frac{\mathrm{d}\Theta}{\mathrm{d}x_3}\right)^2\right)\right] = -P_3$$

with

$$P_3 = \int_{s_-}^{s_+}\int_{-1}^{1} f_3 \,\mathrm{d}r\,\mathrm{d}s + \int_{s_-}^{s_+}[g_3^+ - g_3^-]\,\mathrm{d}s$$

and where

$$S = \int_{s_-}^{s_+}\int_{-1}^{1}\mathrm{d}r\,\mathrm{d}s, \quad S_x = \int_{s_-}^{s_+}\int_{-1}^{1} x\,\mathrm{d}r\,\mathrm{d}s,$$

$$S_{\omega_n} = \int_{s_-}^{s_+}\int_{-1}^{1}\omega\,\mathrm{d}r\,\mathrm{d}s, \quad I_x = \int_{s_-}^{s_+}\int_{-1}^{1}\|x\|^2\,\mathrm{d}r\,\mathrm{d}s.$$

Proof. *The reduced problem $\mathscr{P}_4$*

According to the previous results, problem $\mathscr{P}_4$ now reduces to

$$\frac{\partial\tau_n^4}{\partial r} + \frac{\partial\tau_t^3}{\partial s} + \frac{\partial\tau_3^2}{\partial x_3} = -f_3 e_3 \quad \text{in } \Omega_0. \tag{111}$$

Let us integrate Eq. (111) upon a section. With the boundary conditions (83) and (102), we obtain

$$\int_{s_-}^{s_+}\int_{-1}^{1}\frac{\partial\tau_3^2}{\partial x_3}\,\mathrm{d}r\,\mathrm{d}s = -P_3 e_3,$$

where we set $P_3 = \int_{s_-}^{s_+}\int_{-1}^{1} f_3\,\mathrm{d}r\,\mathrm{d}s + \int_{s_-}^{s_+}[g_3^+ - g_3^-]\,\mathrm{d}s$. Then replacing τ_3^2 with its expression of Result 2, we obtain the equilibrium equation of traction-compression:

$$\frac{E}{2\mu}\frac{\partial}{\partial x_3}\left[\int_{s_-}^{s_+}\int_{-1}^{1}\left\|\frac{\partial U^1}{\partial x_3}\right\|^2 + 2\frac{\partial u_3^2}{\partial x_3}\,\mathrm{d}r\,\mathrm{d}s\right] = -P_3.$$

As

$$\frac{\partial u_3^2}{\partial x_3} = \frac{\mathrm{d}u_3}{\mathrm{d}x_3} - \overline{x}\,\overline{\frac{\mathrm{d}R_\Theta}{\mathrm{d}x_3}}\frac{\mathrm{d}V}{\mathrm{d}x_3} - \overline{x}\,\overline{R_\Theta}\frac{\mathrm{d}^2V}{\mathrm{d}x_3^2} - \omega_n\frac{\mathrm{d}^2\Theta}{\mathrm{d}x_3^2}$$

and

$$\frac{\mathrm{d}R_\Theta}{\mathrm{d}x_3} = \Lambda R_\Theta\frac{\mathrm{d}\Theta}{\mathrm{d}x_3}$$

we get

$$\frac{\partial u_3^2}{\partial x_3} = \frac{\mathrm{d}u_3}{\mathrm{d}x_3} - \overline{x}\,\overline{R_\Theta}\frac{\mathrm{d}^2V}{\mathrm{d}x_3^2} - \overline{x}\,\overline{\Lambda R_\Theta}\frac{\mathrm{d}V}{\mathrm{d}x_3}\frac{\mathrm{d}\Theta}{\mathrm{d}x_3} - \omega_n\frac{\mathrm{d}^2\Theta}{\mathrm{d}x_3^2}.$$

In the same way, we have

$$\frac{\partial U^1}{\partial x_3} = \frac{\mathrm{d}V}{\mathrm{d}x_3} + \Lambda R_\Theta x\frac{\mathrm{d}\Theta}{\mathrm{d}x_3}.$$

As

$$\left\|\frac{\partial U^1}{\partial x_3}\right\|^2 = \overline{\frac{\partial U^1}{\partial x_3}}\frac{\partial U^1}{\partial x_3}$$

we get

$$\left\|\frac{\partial U^1}{\partial x_3}\right\|^2 = \left\|\frac{\mathrm{d}V}{\mathrm{d}x_3}\right\|^2 + 2\overline{\Lambda R_\Theta x}\frac{\mathrm{d}V}{\mathrm{d}x_3}\frac{\mathrm{d}\Theta}{\mathrm{d}x_3} + \|x\|^2\left(\frac{\mathrm{d}\Theta}{\mathrm{d}x_3}\right)^2$$

and the traction equilibrium equation reduces to

$$\frac{E}{\mu}\frac{\mathrm{d}}{\mathrm{d}x_3}\left[S\frac{\mathrm{d}u_3}{\mathrm{d}x_3} - \overline{R_\Theta S_x}\frac{\mathrm{d}^2V}{\mathrm{d}x_3^2} - S_{\omega_n}\frac{\mathrm{d}^2\Theta}{\mathrm{d}x_3^2} + \frac{1}{2}\left(S\left\|\frac{\mathrm{d}V}{\mathrm{d}x_3}\right\|^2 + I_x\left(\frac{\mathrm{d}\Theta}{\mathrm{d}x_3}\right)^2\right)\right] = -P_3$$

with

$$S = \int_{s_-}^{s_+}\int_{-1}^{1} \mathrm{d}r\,\mathrm{d}s, \quad S_x = \int_{s_-}^{s_+}\int_{-1}^{1} x\,\mathrm{d}r\,\mathrm{d}s,$$

$$S_{\omega_n} = \int_{s_-}^{s_+}\int_{-1}^{1} \omega_n\,\mathrm{d}r\,\mathrm{d}s, \quad I_x = \int_{s_-}^{s_+}\int_{-1}^{1} \|x\|^2\,\mathrm{d}r\,\mathrm{d}s.$$

which ends the proof of Result 3. □

Result 4 (*Twist equation*). *For a level of applied forces such as* $\mathscr{F}_t=\mathscr{F}_n=\varepsilon^5$, $\mathscr{G}_t=\mathscr{G}_n=\varepsilon^5$ *and* $\mathscr{F}_3=\mathscr{G}_3=\varepsilon^4$, *the dimensionless unknowns of the displacement field* $V(x_3)$, $u_3(x_3)$ *and* $\Theta(x_3)$ *are solution of the following twist equation*:

$$\frac{\mathrm{d}}{\mathrm{d}x_3}\left[\int_{s_-}^{s_+}\int_{-1}^{1} -r(1-c_0Q)\sigma_{t3}^2 + \omega_N\frac{\partial\sigma_{33}^2}{\partial x_3} + \sigma_{33}^2\overline{\Lambda\psi^1}\frac{\partial\psi^1}{\partial x_3}\,\mathrm{d}r\,\mathrm{d}s\right] = -M_t - \frac{\mathrm{d}M_3^t}{\mathrm{d}x_3},$$

where

$$M_3^t = \int_{s_-}^{s_+}\int_{-1}^{1} \omega_N f_3\,\mathrm{d}r\,\mathrm{d}s + \int_{s_-}^{s_+} \omega_N[g_3^+ - g_3^-]\,\mathrm{d}s,$$

$$M_t = -\int_{s_-}^{s_+} \overline{\Lambda\psi^1}\left(\int_{-1}^{1} f\,\mathrm{d}r + [g^+ - g^-]\right)\mathrm{d}s.$$

We recall that

$$\sigma_{t3}^2 = -2r\frac{\mathrm{d}\Theta}{\mathrm{d}x_3},$$

$$\sigma_{33}^2 = \frac{E}{\mu}\frac{\partial u_3^2}{\partial x_3} + \frac{E}{2\mu}\left\|\frac{\partial\psi^1}{\partial x_3}\right\|^2,$$

$$\psi^1 = U^1 + x$$

with

$$\left|\begin{array}{l} U^1 = V(x_3) + (R_\Theta - I_2)x \\ u_3^2 = u_3(x_3) - \overline{R_\Theta x}\dfrac{\mathrm{d}V}{\mathrm{d}x_3} - \omega_n\dfrac{\mathrm{d}\Theta}{\mathrm{d}x_3} \end{array}\right.$$

and

$$Q = q + \bar{V}N,$$

$$\omega_n = \omega_N + \bar{V}\Lambda N R_\theta x,$$

where

$$R_\Theta = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Proof. In a first time, we will establish a relation which will be useful to obtain the twist and bending equations.

Problem $\mathscr{P}_5$:

The cancellation of the factor of ε^5 leads to problem $\mathscr{P}_5$ which writes

$$\frac{\partial\tau_n^5}{\partial r} + \frac{\partial\tau_t^4}{\partial s} - c_0\tau_n^4 + rc_0\frac{\partial\tau_t^3}{\partial s} + \frac{\partial\tau_3^3}{\partial x_3} = -f \quad \text{in } \Omega_0, \tag{112}$$

$$\tau_n^5 = g^\pm \quad \text{for } r = \pm 1,$$
$$\tau_t^5 = 0 \quad \text{for } s = s_- \text{ and } s = s_+. \tag{113}$$

Using (113), the integration of Eq. (112) with respect to r, after projection onto the plane of a section, leads to

$$\int_{-1}^{1} \Pi\left(\frac{\partial\tau_t^4}{\partial s} - c_0\tau_n^4 + rc_0\frac{\partial\tau_t^3}{\partial s} + \frac{\partial\tau_3^3}{\partial x_3}\right)\mathrm{d}r = -\int_{-1}^{1} \Pi f\,\mathrm{d}r - \Pi[g^+ - g^-], \tag{114}$$

where we recall that $\Pi = (I - e_3 \otimes e_3)$. On the other hand, let us multiply Eq. (111) by r and integrate it with respect to r after projection onto the plane of a section. We get

$$\int_{-1}^{1} \Pi\left(r\frac{\partial\tau_n^4}{\partial r} + r\frac{\partial\tau_t^3}{\partial s} + r\frac{\partial\tau_3^2}{\partial x_3}\right)\mathrm{d}r = 0.$$

Using (102), an integration by parts of the first term leads to

$$\int_{-1}^{1} \Pi\left(-\tau_n^4 + r\frac{\partial\tau_t^3}{\partial s} + r\frac{\partial\tau_3^2}{\partial x_3}\right)\mathrm{d}r = 0. \tag{115}$$

Let us now multiply the previous equation with $c_0(s)$ and replace it in (114). We obtain

$$\int_{-1}^{1} \Pi\left(\frac{\partial\tau_t^4}{\partial s} + \frac{\partial\tau_3^3}{\partial x_3} - rc_0\frac{\partial\tau_3^2}{\partial x_3}\right)\mathrm{d}r = -p, \tag{116}$$

where we set

$$p = \int_{-1}^{1} \Pi f\,\mathrm{d}r + \Pi[g^+ - g^-].$$

This last equation will be used to establish the twist and bending equilibrium equations. To obtain the twist equation, we let appear explicitly a torque in the equations. To do this, let us multiply Eq. (116) with $\Lambda\psi^1$, and integrate the result with respect to s. We obtain

$$\int_{s_-}^{s_+}\int_{-1}^{1} \overline{\Lambda\psi^1}\Pi\left(\frac{\partial\tau_t^4}{\partial s} + \frac{\partial\tau_3^3}{\partial x_3} - rc_0\frac{\partial\tau_3^2}{\partial x_3}\right)\mathrm{d}r\,\mathrm{d}s = -\int_{s_-}^{s_+} \overline{\Lambda\psi^1}p\,\mathrm{d}s. \tag{117}$$

As we have

$$\overline{\Lambda\psi^1}\Pi\frac{\partial\tau_t^4}{\partial s} = \frac{\partial\overline{\Lambda\psi^1}\Pi\tau_t^4}{\partial s} - \frac{\partial\overline{\Lambda\psi^1}}{\partial s}\Pi\tau_t^4$$

and

$$\frac{\partial \Lambda\psi^1}{\partial s} = \Lambda \frac{\partial \psi^1}{\partial s} = \Lambda T = N$$

we get

$$\overline{\Lambda\psi^1} \Pi \frac{\partial \tau_t^4}{\partial s} = \frac{\partial \overline{\Lambda\psi^1} \Pi \tau_t^4}{\partial s} - \tau_{tN}^4.$$

Using the last relation and the boundary condition (103), Eq. (117) becomes

$$\int_{s_-}^{s_+} \int_{-1}^{1} \left(-\tau_{tN}^4 + \overline{\Lambda\psi^1} \Pi \frac{\partial \tau_3^3}{\partial x_3} - r c_0 \overline{\Lambda\psi^1} \Pi \frac{\partial \tau_3^2}{\partial x_3} \right) \times \mathrm{d}r\, \mathrm{d}s = -M_t, \tag{118}$$

where we set

$$M_t = \int_{s_-}^{s_+} \overline{\Lambda\psi^1}\, p\, \mathrm{d}s.$$

Now, to simplify Eq. (118), we use the following relation established in Appendix A:

$$\int_{-1}^{1} \tau_{tN}^4\, \mathrm{d}r = \int_{-1}^{1} \left(\tau_{nT}^4 + \sigma_{t3}^3 \overline{\frac{\partial \psi^1}{\partial x_3}} N \right) \mathrm{d}r. \tag{119}$$

Using the previous relation, Eq. (118) can be written as

$$\int_{s_-}^{s_+} \int_{-1}^{1} \left(-\tau_{nT}^4 - \sigma_{t3}^3 \overline{\frac{\partial \psi^1}{\partial x_3}} N + \overline{\Lambda\psi^1} \Pi \frac{\partial \tau_3^3}{\partial x_3} - r c_0 \overline{\Lambda\psi^1} \Pi \frac{\partial \tau_3^2}{\partial x_3} \right) \mathrm{d}r\, \mathrm{d}s = -M_t. \tag{120}$$

Now let us explain every term of Eq. (120) with respect to the data of the problem.

To obtain the expression of τ_{nT}^4, let us project Eq. (115) of problem $\mathscr{P}_4$ onto T. We get

$$\int_{-1}^{1} \left(-\tau_{nT}^4 + r \frac{\partial \tau_{tT}^3}{\partial s} - r c_0 \tau_{tN}^3 + r \frac{\partial \tau_{3T}^2}{\partial x_3} \right) \mathrm{d}r = 0.$$

Using the boundary condition (83), the integration of this equation with respect to s leads to

$$\int_{s_-}^{s_+} \int_{-1}^{1} \left(-\tau_{nT}^4 - r c_0 \tau_{tN}^3 + r \frac{\partial \tau_{3T}^2}{\partial x_3} \right) \mathrm{d}r\, \mathrm{d}s = 0. \tag{121}$$

On the other hand, as we proved that $\sigma_{tt}^2 = 0$ and $\tau_n^3 = 0$, expression (130) of τ_{tN}^3 established in Appendix A reduces to

$$\tau_t N^3 = \sigma_{t3}^2 \overline{\frac{\partial \psi^1}{\partial x_3}} N.$$

This last relation enables to write Eq. (121) as follows

$$\int_{s_-}^{s_+} \int_{-1}^{1} \tau_{nT}^4\, \mathrm{d}r\, \mathrm{d}s = \int_{s_-}^{s_+} \int_{-1}^{1} \left(-r c_0 \sigma_{t3}^2 \overline{\frac{\partial \psi^1}{\partial x_3}} N + r \frac{\partial \tau_{3T}^2}{\partial x_3} \right) \mathrm{d}r\, \mathrm{d}s. \tag{122}$$

Thus we have an explicit expression for the first term of (120). Let us now explain the other terms depending on $\Lambda\psi^1$. First we have

$$\overline{\Lambda\psi^1} \Pi \frac{\partial \tau_3^2}{\partial x_3} = \frac{\partial}{\partial x_3} \left[\overline{\Lambda\psi^1} \Pi \tau_3^2 \right] - \frac{\partial \overline{\Lambda\psi^1}}{\partial x_3} \Pi \tau_3^2.$$

As $\Pi \tau_3^2 = \tau_{3T}^2 T = \sigma_{t3}^2 T$, we easily obtain

$$\overline{\frac{\partial \Lambda\psi^1}{\partial x_3}} \Pi \tau_3^2 = \sigma_{t3}^2 \overline{\frac{\partial \psi^1}{\partial x_3}} \overline{\Lambda} T = -\sigma_{t3}^2 \overline{\frac{\partial \psi^1}{\partial x_3}} N$$

so that

$$\overline{\Lambda\psi^1} \Pi \frac{\partial \tau_3^2}{\partial x_3} = \frac{\partial}{\partial x_3} \left[\overline{\Lambda\psi^1} \Pi \tau_3^2 \right] + \sigma_{t3}^2 \overline{\frac{\partial \psi^1}{\partial x_3}} N. \tag{123}$$

To explain the term involving τ_3^3, we use the following relation whose proof is detailed in Appendix A:

$$\Pi \tau_3^3 = \tau_{t3}^3 T + \sigma_{t3}^2 \Psi N + \sigma_{33}^2 \frac{\partial \psi^1}{\partial x_3}. \tag{124}$$

Using relation (124), as

$$\overline{\Lambda\psi^1} \Pi \frac{\partial \tau_3^3}{\partial x_3} = \frac{\partial}{\partial x_3} \left(\overline{\Lambda\psi^1} \Pi \tau_3^3 \right) - \overline{\frac{\partial \Lambda\psi^1}{\partial x_3}} \Pi \tau_3^3$$

we have

$$\overline{\Lambda\psi^1} \Pi \frac{\partial \tau_3^3}{\partial x_3} = \frac{\partial}{\partial x_3} \left(\overline{\Lambda\psi^1} \Pi \tau_3^3 \right) + \tau_{t3}^3 \overline{\frac{\partial \psi^1}{\partial x_3}} N - \sigma_{t3}^2 \overline{\frac{\partial \psi^1}{\partial x_3}} \Psi T.$$

Let us replace (122) and (123) in (117). As $\sigma_{t3}^2 = \tau_{t3}^2$ in linear with respect to r according to (109), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x_3} \left[\int_{s_-}^{s_+} \int_{-1}^{1} \left(-r \sigma_{t3}^2 - r c_0 \overline{\Lambda\psi^1} T \tau_{3T}^2 + \overline{\Lambda\psi^1} \Pi \tau_3^3 \right) \mathrm{d}r\, \mathrm{d}s \right] = -M_t.$$

As $\psi^1 = HT + QN$, the last equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}x_3} \left[\int_{s_-}^{s_+} \int_{-1}^{1} \left(-r(1 - c_0 Q) \sigma_{t3}^2 - \sigma_{t3}^3 Q + \sigma_{33}^2 \overline{\Lambda\psi^1} \frac{\partial \psi^1}{\partial x_3} \right) \mathrm{d}r\, \mathrm{d}s \right] = -M_t. \tag{125}$$

To explain the expression of $\sigma_{t3}^3 Q$, let us start again from Eq. (111) of problem $\mathscr{P}_4$, which is multiplied with ω_N and projected onto e_3. An integration with respect to r and s then leads to

$$\int_{s_-}^{s_+} \int_{-1}^{1} \left(\omega_N \frac{\partial \tau_{n3}^4}{\partial r} + \omega_N \frac{\partial \tau_{t3}^3}{\partial s} + \omega_N \frac{\partial \tau_{33}^2}{\partial x_3} \right) \mathrm{d}r\, \mathrm{d}s = -\int_{s_-}^{s_+} \int_{-1}^{1} \omega_N f_3\, \mathrm{d}r\, \mathrm{d}s - \int_{s_-}^{s_+} \omega_N [g_3^+ - g_3^-]\, \mathrm{d}s.$$

Using the boundary conditions (83) and (102), and integrating by parts the second term, we obtain

$$\int_{s_-}^{s_+}\int_{-1}^{1}\left(Q\sigma_{t3}^3+\omega_N\frac{\partial\sigma_{33}^2}{\partial x_3}\right)\mathrm{d}r\,\mathrm{d}s=-M_3^t$$

with

$$M_3^t=\int_{s_-}^{s_+}\int_{-1}^{1}\omega_N f_3\,\mathrm{d}r\,\mathrm{d}s+\int_{s_-}^{s_+}\omega_N\left[g_3^+-g_3^-\right]\mathrm{d}s.$$

Finally from (125), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x_3}\left[\int_{s_-}^{s_+}\int_{-1}^{1}\left(-r(1-c_0Q)\sigma_{t3}^2+\omega_N\frac{\partial\sigma_{33}^2}{\partial x_3}\right.\right.$$
$$\left.\left.+\sigma_{33}^2\overline{\Lambda\psi^1}\frac{\partial\psi^1}{\partial x_3}\right)\mathrm{d}r\,\mathrm{d}s\right]=-M_t-\frac{\mathrm{d}M_3^t}{\mathrm{d}x_3}$$

which corresponds to the twist equation of Result 4. □

Result 5 (*Bending equations*). *For a level of applied forces such as $\mathscr{F}_t=\mathscr{F}_n=\varepsilon^5$, $\mathscr{G}_t=\mathscr{G}_n=\varepsilon^5$ and $\mathscr{F}_3=\mathscr{G}_3=\varepsilon^4$, the dimensionless unknowns of the displacement field $V(x_3)$, $u_3(x_3)$ and $\Theta(x_3)$ are solution of the following bending equations*:

$$\frac{E}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}\left[\int_{s_-}^{s_+}\int_{-1}^{1}\left[\left\|\frac{\partial U^1}{\partial x_3}\right\|^2+2\frac{\partial u_3^2}{\partial x_3}\right]\psi^1\,\mathrm{d}r\,\mathrm{d}s\right]$$
$$+\frac{\mathrm{d}}{\mathrm{d}x_3}\left[\int_{s_-}^{s_+}\int_{-1}^{1}2r^2c_0\frac{\mathrm{d}\Theta}{\mathrm{d}x_3}T\,\mathrm{d}r\,\mathrm{d}s\right]=-P-\frac{\mathrm{d}M_3^f}{\mathrm{d}x_3},$$

where

$$P=\int_{s_-}^{s_+}\int_{-1}^{1}\Pi f\,\mathrm{d}r\,\mathrm{d}s+\int_{s_-}^{s_+}\Pi\left[g^+-g^-\right]\mathrm{d}s,$$
$$M_3^f=\int_{s_-}^{s_+}\int_{-1}^{1}\psi^1 f_3\,\mathrm{d}r\,\mathrm{d}s+\int_{s_-}^{s_+}\psi^1\left[g_3^+-g_3^-\right]\mathrm{d}s$$

and

$$\psi^1=U^1+x$$

with

$$\left|\begin{array}{l}U^1=V(x_3)+(R_\Theta-I_2)x,\\ u_3^2=u_3(x_3)-\overline{R_\Theta x}\dfrac{\mathrm{d}V}{\mathrm{d}x_3}-\omega_n\dfrac{\mathrm{d}\Theta}{\mathrm{d}x_3}.\end{array}\right.$$

Proof. The proof of this result starts from (116) which is integrated with respect to s. Using the boundary condition (103), we get

$$\int_{s_-}^{s_+}\int_{-1}^{1}\Pi\left(\frac{\partial\tau_3^3}{\partial x_3}-rc_0\frac{\partial\tau_3^2}{\partial x_3}\right)\mathrm{d}r\,\mathrm{d}s=-P$$

with $P=\int_{s_-}^{s_+}p\,\mathrm{d}s$.

This equation can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}x_3}\left\{\int_{s_-}^{s_+}\int_{-1}^{1}\Pi\left(\tau_3^3-rc_0\tau_3^2\right)\mathrm{d}r\,\mathrm{d}s\right\}=-P.$$

As we have already proved at (124) that

$$\Pi\tau_3^3=\sigma_{t3}^3T+\sigma_{t3}^2\Psi+\sigma_{33}^2\frac{\partial\psi^1}{\partial x_3}$$

we obtain (σ_{t3}^2 being linear with respect to r)

$$\frac{\mathrm{d}}{\mathrm{d}x_3}\left\{\int_{s_-}^{s_+}\int_{-1}^{1}\left(\sigma_{13}^3T+\sigma_{33}^2\frac{\partial\psi^3}{\partial x_3}-rc\Pi\tau_3^2\right)\mathrm{d}r\,\mathrm{d}s\right\}=-P. \quad (126)$$

The last unknown term to explain is σ_{t3}^3T. To do this, let us use again Eq. (111) of problem $\mathscr{P}_4$ which is projected upon e_3. An integration upon the thickness then leads to

$$\int_{-1}^{1}\left(\frac{\partial\tau_{t3}^3}{\partial s}+\frac{\partial\tau_{33}^2}{\partial x_3}\right)\mathrm{d}r=-\int_{-1}^{1}f_3\,\mathrm{d}r-\left[g_3^+-g_3^-\right].$$

Now let us multiply the last equation with ψ^1 and integrate the result with respect to s. We get

$$\int_{s_-}^{s_+}\int_{-1}^{1}\left(\psi^1\frac{\partial\sigma_{t3}^3}{\partial s}+\psi^1\frac{\partial\sigma_{33}^2}{\partial x_3}\right)\mathrm{d}r\,\mathrm{d}s$$
$$=-\int_{s_-}^{s_+}\int_{-1}^{1}\psi^1f_3\,\mathrm{d}r\,\mathrm{d}s-\int_{s_-}^{s_+}\psi^1\left[g_3^+-g_3^-\right]\mathrm{d}s$$

whose first term is then integrated by parts. Using (83), we obtain

$$\int_{s_-}^{s_+}\int_{-1}^{1}\left(-\sigma_{t3}^3T+\psi^1\frac{\partial\sigma_{33}^2}{\partial x_3}\right)\mathrm{d}r\,\mathrm{d}s=-M_3^f,$$

where we set

$$M_3^f=\int_{s_-}^{s_+}\int_{-1}^{1}\psi^1f_3\,\mathrm{d}r\,\mathrm{d}s+\int_{s_-}^{s_+}\psi^1\left[g_3^+-g_3^-\right]\mathrm{d}s.$$

Replacing this last expression in (126), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x_3}\left\{\int_{s_-}^{s_+}\int_{-1}^{1}\psi^1\frac{\partial\sigma_{33}^2}{\partial x_3}\mathrm{d}r\,\mathrm{d}s\right.$$
$$\left.+\int_{s_-}^{s_+}\int_{-1}^{1}\left(\sigma_{33}^2\frac{\partial\psi^1}{\partial x_3}-rc_0\sigma_{t3}^2T\right)\mathrm{d}r\,\mathrm{d}s\right\}$$
$$=-P-\frac{\mathrm{d}M_3^f}{\mathrm{d}x_3}$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}x_3}\left\{\int_{s_-}^{s_+}\int_{-1}^{1}\left(\frac{\partial\sigma_{33}^2\psi^1}{\partial x_3}-rc_0\sigma_{t3}^2T\right)\mathrm{d}r\,\mathrm{d}s\right\}$$
$$=-P-\frac{\mathrm{d}M_3}{\mathrm{d}x_3}.$$

Finally substituting σ^2_{33} and σ^2_{t3} with their expressions of Result 2, we obtain

$$\frac{E}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}\left[\int_{s_-}^{s_+}\int_{-1}^{1}\left[\left\|\frac{\partial U^1}{\partial x_3}\right\|^2+2\overline{\frac{\partial u_3^2}{\partial x_3}}\right]\psi^1\,\mathrm{d}r\,\mathrm{d}s\right]$$
$$+\frac{\mathrm{d}}{\mathrm{d}x_3}\left[\int_{s_-}^{s_+}\int_{-1}^{1}2r^2c_0\frac{\mathrm{d}\Theta}{\mathrm{d}x_3}T\,\mathrm{d}r\,\mathrm{d}s\right]=-P-\frac{\mathrm{d}M_3}{\mathrm{d}x_3}$$

which constitutes the bending equations of Result 5. □

6. Conclusion

In this paper, we deduced a non-linear model for thin-walled rods with strongly curved profile, from asymptotic expansion of the non-linear three-dimensional equations. With the constructive approach developed, based on a dimensional analysis of equilibrium equations, the order of magnitude of the displacements are directly deduced from the level of applied forces.

The non-linear kinematics obtained in Result 1 generalize that of Vlassov for large or moderate rotations. It is generally obtained in the literature by making a priori assumptions similar to Vlassov ones [3–8,10]. In contrary, in this paper these non-linear kinematics are justified rigorously by asymptotic expansions for the level of applied forces considered ; the levels of forces being characterized by the non-dimensional numbers naturally introduced.

The non-linear one-dimensional traction, twist and bending equations of Results (3)–(5) are not classical and do not seem to have any equivalent in the literature. They constitute a system of four non-linear differential equations strongly coupled, which can be expressed in terms of the four unknowns[7] $V(x_3)$, $u_3(x_3)$, $\Theta(x_3)$. In particular, the twist equation contains cubic terms with respect to Θ as in the model of Gobarah and Tso [4] derived from a priori assumptions. Moreover the twist equation is coupled with the traction (and the bending) equations. This coupling characterizes the shortening effect observed experimentally for large rotations.

Appendix A. Proof of some intermediary formula

In this appendix we detail the proof of the intermediary (88) (107) (119) (124) used in the calculations of the previous results.

First, using relations (30) and (31), we have

$$\tau_t^2=\sigma_{tt}^2T+\sigma_{tt}^1\frac{\partial\psi^2}{\partial s}+\sigma_{tn}^2N+\sigma_{t3}^2e_3,$$

$$\tau_n^2=\sigma_{tn}^2T+\sigma_{nn}^2N+\sigma_{n3}^2e_3$$

and as a consequence

$$\tau_{tN}^2=\tau_{nT}^2+\sigma_{tt}^1\overline{\frac{\partial U^2}{\partial s}}N. \tag{127}$$

[7] The displacement $V(x_3)$ has two components in the plane of a section.

As $\sigma^1_{tt}=\overline{\tau^1_t}T$ is linear with respect to r according to (77), we deduce that

$$\int_{-1}^{1}\tau_{tN}^2\,\mathrm{d}r=\int_{-1}^{1}\tau_{nT}^2\,\mathrm{d}r$$

which concludes the proof of (88).

Let us now establish (107) used for the proof of Result (2) and which writes

$$\int_{-1}^{1}\tau_{tN}^3\,\mathrm{d}r=\int_{-1}^{1}\tau_{nT}^3\,\mathrm{d}r.$$

According to the previous calculations, we have

$$\tau_t^3=\sigma_{tt}^3T+\sigma_{tt}^2\frac{\partial U^2}{\partial s}+\sigma_{tn}^3N+\sigma_{t3}^3e_3+\sigma_{t3}^2\frac{\partial\psi^1}{\partial x_3}, \tag{128}$$

$$\tau_n^3=\sigma_{tn}^3T+\sigma_{nn}^3N+\sigma_{n3}^3e_3, \tag{129}$$

so that

$$\tau_{tN}^3=\tau_{nT}^3+\sigma_{tt}^2\overline{\frac{\partial U^2}{\partial s}}N+\sigma_{t3}^2\overline{\frac{\partial\psi^1}{\partial x_3}}N. \tag{130}$$

According to expression (97) of τ^2_t, the components $\sigma^2_{tt}=\overline{\tau^2_t}T$ and $\sigma^2_{t3}=\overline{\tau^2_t}e_3$ are linear with respect to r and we get

$$\int_{-1}^{1}\tau_{tN}^3\,\mathrm{d}r=\int_{-1}^{1}\tau_{nT}^3\,\mathrm{d}r$$

which concludes the proof of (107).

The proof of relation (119)

$$\int_{-1}^{1}\tau_{tN}^4\,\mathrm{d}r=\int_{-1}^{1}\left(\tau_{nT}^4+\sigma_{t3}^3\overline{\frac{\partial\psi^1}{\partial x_3}}N\right)\mathrm{d}r$$

can be made in a similar way. Indeed, using relations (30) and (31), we have

$$\tau_n^4=\sigma_{tn}^4T+\sigma_{nn}^4N+\sigma_{n3}^4e_3,$$

$$\tau_t^4=\sigma_{tt}^4T+\sigma_{tt}^3\frac{\partial\psi^2}{\partial s}+rc_0\sigma_{tt}^3T+\sigma_{tn}^4N+\sigma_{t3}^4e_3$$
$$+\sigma_{t3}^3\frac{\partial\psi^1}{\partial x_3}+\sigma_{t3}^2\frac{\partial\psi^2}{\partial x_3}.$$

As $\partial\psi^2/\partial s=\partial U^2/\partial s-rc_0T$ according to (67), we have

$$\tau_{tN}^4=\sigma_{tn}^4+\sigma_{tt}^3\overline{\frac{\partial U^2}{\partial s}}N+\sigma_{t3}^3\overline{\frac{\partial\psi^1}{\partial x_3}}N+\sigma_{t3}^2\overline{\frac{\partial\psi^2}{\partial x_3}}N.$$

Moreover, as $\left(\overline{\partial U^2/\partial s}\right)N=\Psi$ according to (108), we get

$$\tau_{tN}^4=\sigma_{tn}^4+\sigma_{tt}^3\Psi+\sigma_{t3}^3\overline{\frac{\partial\psi^1}{\partial x_3}}N+\sigma_{t3}^2\overline{\frac{\partial U^2}{\partial x_3}}N.$$

Now let us integrate with respect to r the previous equation. We obtain

$$\int_{-1}^{1} \tau_{tN}^4 \, \mathrm{d}r = \int_{-1}^{1} \left(\tau_{nT}^4 + \sigma_{tt}^3 \Psi + \sigma_{t3}^3 \overline{\frac{\partial \psi^1}{\partial x_3}} N + \sigma_{t3}^2 \overline{\frac{\partial U^2}{\partial x_3}} N \right) \mathrm{d}r.$$

According to (109) (or to Result 2), σ_{t3}^2 is linear with respect to r. On the other hand, the previous results and in particular (128) leads to $\sigma_{tt}^3 = \overline{\tau_t^3} T$. Moreover using the boundary condition (83) and Eq. (111), we get $\int_{-1}^{1} \sigma_{tt}^3 \, \mathrm{d}r = 0$. So that the last equation reduces to

$$\int_{-1}^{1} \tau_{tN}^4 \, \mathrm{d}r = \int_{-1}^{1} \left(\tau_{nT}^4 + \sigma_{t3}^3 \overline{\frac{\partial \psi^1}{\partial x_3}} N \right) \mathrm{d}r$$

which concludes the proof of (119).

To finish let us established relations (124) used in the proof of Result 4:

$$\Pi \tau_3^3 = \tau_{t3}^3 T + \sigma_{t3}^2 \Psi N + \sigma_{33}^2 \frac{\partial \psi^1}{\partial x_3}.$$

Using relations (30) and (31), we have

$$\tau_3^3 = \sigma_{t3}^3 T + \sigma_{t3}^2 \frac{\partial \psi^2}{\partial s} + r c_0 \sigma_{t3}^2 T + \sigma_{33}^3 e_3 + \sigma_{33}^2 \frac{\partial \psi^1}{\partial x_3}.$$

Projecting this relation onto the plane of a section and replacing ψ^2 with its expression (67), we obtain

$$\Pi \tau_3^3 = \sigma_{t3}^3 T + \sigma_{t3}^2 \Pi \frac{\partial U^2}{\partial s} + \sigma_{33}^2 \Pi \frac{\partial \psi^1}{\partial x_3}.$$

On the first hand, we have $\Pi(\partial U^2/\partial s) = \Psi N$ because $\overline{(\partial U^2/\partial s)} N = \Psi$ according to (108). Moreover $\overline{\partial U^2/\partial s T} = 0$ according to (84). Thus we get

$$\Pi \tau_3^3 = \sigma_{t3}^3 T + \sigma_{t3}^2 \Psi N + \sigma_{33}^2 \Pi \frac{\partial \psi^1}{\partial x_3}.$$

On the other hand, we have

$$\tau_t^3 = \sigma_{tt}^3 T + \sigma_{t3}^3 e_3 + \sigma_{t3}^2 \frac{\partial \psi^1}{\partial x_3}.$$

Thus $\tau_{t3}^3 = \sigma_{t3}^3$ and

$$\Pi \tau_3^3 = \tau_{t3}^3 T + \sigma_{t3}^2 \Psi N + \sigma_{33}^2 \Pi \frac{\partial \psi^1}{\partial x_3}$$

which concludes the proof of relation (124).

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